Exercise 5.1

Question 1:

Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

Answer

The given function is $f(x) = 5x - 3$

At $x = 0, f(0) = 5 \times 0 - 3 = 3$

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0}(5x - 3) = 5 \times 0 - 3 = -3$

$\therefore \lim\limits_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at x = 0

At $x = -3, f(-3) = 5 \times (-3) - 3 = -18$

$\lim\limits_{x \to -3} f(x) = \lim\limits_{x \to -3}(5x - 3) = 5 \times (-3) - 3 = -18$

$\therefore \lim\limits_{x \to -3} f(x) = f(-3)$

Therefore, f is continuous at x = −3

At $x = 5, f(x) = f(5) = 5 \times 5 - 3 = 25 - 3 = 22$

$\lim\limits_{x \to 5} f(x) = \lim\limits_{x \to 5}(5x - 3) = 5 \times 5 - 3 = 22$

$\therefore \lim\limits_{x \to 5} f(x) = f(5)$

Therefore, f is continuous at x = 5

Question 2:

Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Answer

The given function is $f(x) = 2x^2 - 1$

At $x = 3, f(x) = f(3) = 2 \times 3^2 - 1 = 17$

$\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3}(2x^2 - 1) = 2 \times 3^2 - 1 = 17$

$\therefore \lim\limits_{x \to 3} f(x) = f(3)$

Thus, f is continuous at x = 3

Question 3:

Examine the following functions for continuity.

(a) $f(x) = x - 5$

(b) $f(x) = \dfrac{1}{x-5}, x \neq 5$

(c) $f(x) = \dfrac{x^2 - 25}{x+5}, x \neq -5$

(d) $f(x) = |x - 5|$

Answer

(a) The given function is $f(x) = x - 5$

It is evident that f is defined at every real number k and its value at k is $k - 5$.

It is also observed that, $\lim\limits_{x \to k} f(x) = \lim\limits_{x \to k} (x - 5) = k - 5 = f(k)$

$\therefore \lim\limits_{x \to k} f(x) = f(k)$

Hence, f is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x) = \dfrac{1}{x-5}, x \neq 5$

For any real number $k \neq 5$, we obtain

$\lim\limits_{x \to k} f(x) = \lim\limits_{x \to k} \dfrac{1}{x-5} = \dfrac{1}{k-5}$

$\text{Also}, f(k) = \dfrac{1}{k-5} \quad (\text{As } k \neq 5)$

$\therefore \lim\limits_{x \to k} f(x) = f(k)$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(c) The given function is $f(x) = \dfrac{x^2 - 25}{x+5}, x \neq -5$

For any real number $c \neq -5$, we obtain

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{x^2 - 25}{x+5} = \lim_{x \to c} \frac{(x+5)(x-5)}{x+5} = \lim_{x \to c} (x-5) = (c-5)$$

$$\text{Also}, f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(d) The given function is $f(x) = |x-5| = \begin{cases} 5-x, & \text{if } x < 5 \\ x-5, & \text{if } x \geq 5 \end{cases}$

This function *f* is defined at all points of the real line.

Let *c* be a point on a real line. Then, $c < 5$ or $c = 5$ or $c > 5$

Case I: $c < 5$

Then, $f(c) = 5 - c$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (5-x) = 5-c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers less than 5.

Case II : $c = 5$

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5} (5-x) = (5-5) = 0$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5} (x-5) = 0$$

$$\therefore \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

Therefore, *f* is continuous at $x = 5$

Case III: $c > 5$

Then, $f(c) = f(5) = c-5$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x-5) = c-5$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers greater than 5.

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

Question 4:

Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

Answer

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

Then, $\lim\limits_{x \to n} f(n) = \lim\limits_{x \to n} (x^n) = n^n$

$\therefore \lim\limits_{x \to n} f(x) = f(n)$

Therefore, f is continuous at n, where n is a positive integer.

Question 5:

Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Answer

The given function f is $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$

At $x = 0$,

It is evident that f is defined at 0 and its value at 0 is 0.

Then, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x = 0$

$\therefore \lim\limits_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

At $x = 1$,

f is defined at 1 and its value at 1 is 1.

The left hand limit of f at $x = 1$ is,

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(5) = 5$$

$$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

Therefore, *f* is not continuous at *x* = 1

At *x* = 2,

f is defined at 2 and its value at 2 is 5.

$$\text{Then, } \lim_{x \to 2} f(x) = \lim_{x \to 2}(5) = 5$$

$$\therefore \lim_{x \to 2} f(x) = f(2)$$

Therefore, *f* is continuous at *x* = 2

Question 6:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

Answer

$$f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

It is evident that the given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line. Then, three cases arise.

(i) *c* < 2

(ii) *c* > 2

(iii) *c* = 2

Case (i) *c* < 2

$$\text{Then, } f(c) = 2c+3$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(2x+3) = 2c+3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < 2

Case (ii) *c* > 2

Then, $f(c) = 2c - 3$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (2x - 3) = 2c - 3$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 2

Case (iii) *c* = 2

Then, the left hand limit of *f* at *x* = 2 is,

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (2x + 3) = 2 \times 2 + 3 = 7$

The right hand limit of *f* at *x* = 2 is,

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (2x - 3) = 2 \times 2 - 3 = 1$

It is observed that the left and right hand limit of *f* at *x* = 2 do not coincide.

Therefore, *f* is not continuous at *x* = 2

Hence, *x* = 2 is the only point of discontinuity of *f*.

Question 7:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} |x| + 3, \text{if } x \le -3 \\ -2x, \text{if } -3 < x < 3 \\ 6x + 2, \text{if } x \ge 3 \end{cases}$$

Answer

$$f(x) = \begin{cases} |x| + 3 = -x + 3, \text{ if } x \le -3 \\ -2x, \text{ if } -3 < x < 3 \\ 6x + 2, \text{ if } x \ge 3 \end{cases}$$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c < -3$, then $f(c) = -c + 3$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (-x + 3) = -c + 3$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* < −3

Case II:

If $c = -3$, then $f(-3) = -(-3) + 3 = 6$

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (-x + 3) = -(-3) + 3 = 6$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x) = -2 \times (-3) = 6$$

$$\therefore \lim_{x \to -3} f(x) = f(-3)$$

Therefore, *f* is continuous at *x* = −3

Case III:

If $-3 < c < 3$, then $f(c) = -2c$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (-2x) = -2c$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous in (−3, 3).

Case IV:

If *c* = 3, then the left hand limit of *f* at *x* = 3 is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x) = -2 \times 3 = -6$$

The right hand limit of *f* at *x* = 3 is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2) = 6 \times 3 + 2 = 20$$

It is observed that the left and right hand limit of *f* at *x* = 3 do not coincide.

Therefore, *f* is not continuous at *x* = 3

Case V:

If $c > 3$, then $f(c) = 6c + 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (6x + 2) = 6c + 2$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 3

Hence, *x* = 3 is the only point of discontinuity of *f*.

Question 8:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Answer

$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

It is known that, $x < 0 \Rightarrow |x| = -x$ and $x > 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \dfrac{|x|}{x} = \dfrac{-x}{x} = -1 & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \dfrac{|x|}{x} = \dfrac{x}{x} = 1, & \text{if } x > 0 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = -1$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (-1) = -1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points $x < 0$

Case II:

If $c = 0$, then the left hand limit of f at $x = 0$ is,

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-1) = -1$

The right hand limit of f at $x = 0$ is,

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (1) = 1$

It is observed that the left and right hand limit of f at $x = 0$ do not coincide.

Therefore, f is not continuous at $x = 0$

Case III:

If $c > 0$, then $f(c) = 1$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (1) = 1$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 0$

Hence, $x = 0$ is the only point of discontinuity of f.

Question 9:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

Answer

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

It is known that, $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \dfrac{x}{|x|} = \dfrac{x}{-x} = -1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

$\Rightarrow f(x) = -1$ for all $x \in \mathbf{R}$

Let c be any real number. Then, $\displaystyle \lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1$

Also, $f(c) = -1 = \displaystyle \lim_{x \to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question 10:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2 +1, & \text{if } x < 1 \end{cases}$$

Answer

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2 +1, & \text{if } x < 1 \end{cases}$$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^2 +1$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x^2 +1) = c^2 +1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* < 1

Case II:

If $c = 1$, then $f(c) = f(1) = 1+1 = 2$

The left hand limit of *f* at *x* = 1 is,

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-}(x^2 +1) = 1^2 +1 = 2$

The right hand limit of *f* at *x* = 1 is,

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+}(x+1) = 1+1 = 2$

$\therefore \lim\limits_{x \to 1} f(x) = f(1)$

Therefore, *f* is continuous at *x* = 1

Case III:

If $c > 1$, then $f(c) = c+1$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x+1) = c+1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Hence, the given function *f* has no point of discontinuity.

Question 11:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

Answer

$$f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c < 2$, then $f(c) = c^3 - 3$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (x^3 - 3) = c^3 - 3$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* < 2

Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^3 - 3) = 2^3 - 3 = 5$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x^2 + 1) = 2^2 + 1 = 5$

$\therefore \lim\limits_{x \to 2} f(x) = f(2)$

Therefore, *f* is continuous at *x* = 2

Case III:

If $c > 2$, then $f(c) = c^2 + 1$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (x^2 + 1) = c^2 + 1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 2

Thus, the given function *f* is continuous at every point on the real line.

Hence, *f* has no point of discontinuity.

Question 12:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

Answer

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^{10} - 1$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \left(x^{10} - 1\right) = c^{10} - 1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

If $c = 1$, then the left hand limit of f at $x = 1$ is,

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \left(x^{10} - 1\right) = 1^{10} - 1 = 1 - 1 = 0$

The right hand limit of f at $x = 1$ is,

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \left(x^2\right) = 1^2 = 1$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c^2$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \left(x^2\right) = c^2$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f.

Question 13:

Is the function defined by

$$f(x) = \begin{cases} x+5, & \text{if } x \leq 1 \\ x-5, & \text{if } x > 1 \end{cases}$$

a continuous function?

Answer

The given function is $f(x) = \begin{cases} x+5, & \text{if } x \leq 1 \\ x-5, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c+5$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x+5) = c+5$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

If $c = 1$, then $f(1) = 1+5 = 6$

The left hand limit of f at $x = 1$ is,

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-}(x+5) = 1+5 = 6$

The right hand limit of f at $x = 1$ is,

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+}(x-5) = 1-5 = -4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c-5$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x-5) = c-5$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f.

Question 14:

Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$$

Answer

The given function is $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

The given function is defined at all points of the interval [0, 10].

Let c be a point in the interval [0, 10].

Case I:

If $0 \le c < 1,$ then $f(c) = 3$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(3) = 3$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous in the interval [0, 1).

Case II:

If $c = 1,$ then $f(3) = 3$

The left hand limit of f at $x = 1$ is,

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-}(3) = 3$

The right hand limit of f at $x = 1$ is,

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+}(4) = 4$

It is observed that the left and right hand limits of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $1 < c < 3,$ then $f(c) = 4$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(4) = 4$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (1, 3).

Case IV:

If $c = 3,$ then $f(c) = 5$

The left hand limit of f at $x = 3$ is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4) = 4$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5) = 5$$

It is observed that the left and right hand limits of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

If $3 < c \leq 10$, then $f(c) = 5$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (5) = 5$

$$\lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at $x = 1$ and $x = 3$

Question 15:

Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Answer

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = 2c$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

If $c = 0$, then $f(c) = f(0) = 0$

The left hand limit of *f* at *x* = 0 is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x) = 2 \times 0 = 0$$

The right hand limit of *f* at *x* = 0 is,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (0) = 0$$

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

Therefore, *f* is continuous at *x* = 0

Case III:

If $0 < c < 1$, then $f(x) = 0$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (0) = 0$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points of the interval (0, 1).

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x) = 4 \times 1 = 4$$

It is observed that the left and right hand limits of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case V:

If $c < 1$, then $f(c) = 4c$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (4x) = 4c$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Hence, *f* is not continuous only at *x* = 1

Question 16:

Discuss the continuity of the function *f*, where *f* is defined by

$$f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$$

Answer

$$f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c < -1$, then $f(c) = -2$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (-2) = -2$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < -1$

Case II:

If $c = -1$, then $f(c) = f(-1) = -2$

The left hand limit of f at $x = -1$ is,

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2) = -2$

The right hand limit of f at $x = -1$ is,

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x) = 2 \times (-1) = -2$

$\therefore \lim_{x \to -1} f(x) = f(-1)$

Therefore, f is continuous at $x = -1$

Case III:

If $-1 < c < 1$, then $f(c) = 2c$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval $(-1, 1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 2 \times 1 = 2$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x) = 2 \times 1 = 2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2$$

$$\therefore \lim_{x \to 1} f(x) = f(c)$$

Therefore, f is continuous at $x = 2$

Case V:

If $c > 1$, then $f(c) = 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (2) = 2$

$$\lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

Question 17:

Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$.

Answer

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

If f is continuous at $x = 3$, then

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) \qquad \ldots(1)$$

Also,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax+1) = 3a+1$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx+3) = 3b+3$$

$$f(3) = 3a+1$$

Therefore, from (1), we obtain

$$3a+1 = 3b+3 = 3a+1$$

$$\Rightarrow 3a+1 = 3b+3$$

$$\Rightarrow 3a = 3b+2$$

$$\Rightarrow a = b + \frac{2}{3}$$

Therefore, the required relationship is given by, $a = b + \dfrac{2}{3}$

Question 18:

For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$

continuous at x = 0? What about continuity at x = 1?

Answer

The given function is $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$

If *f* is continuous at x = 0, then

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x \to 0^-} \lambda(x^2 - 2x) = \lim_{x \to 0^+} (4x+1) = \lambda(0^2 - 2 \times 0)$$

$$\Rightarrow \lambda(0^2 - 2 \times 0) = 4 \times 0 + 1 = 0$$

$$\Rightarrow 0 = 1 = 0, \text{ which is not possible}$$

Therefore, there is no value of λ for which *f* is continuous at x = 0

At $x = 1$,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$\lim\limits_{x \to 1} (4x+1) = 4 \times 1 + 1 = 5$

$\therefore \lim\limits_{x \to 1} f(x) = f(1)$

Therefore, for any values of λ, f is continuous at $x = 1$

Question 19:

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to x.

Answer

The given function is $g(x) = x - [x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$g(n) = n - [n] = n - n = 0$

The left hand limit of f at $x = n$ is,

$\lim\limits_{x \to n^-} g(x) = \lim\limits_{x \to n^-} (x - [x]) = \lim\limits_{x \to n^-} (x) - \lim\limits_{x \to n^-} [x] = n - (n-1) = 1$

The right hand limit of f at $x = n$ is,

$\lim\limits_{x \to n^+} g(x) = \lim\limits_{x \to n^+} (x - [x]) = \lim\limits_{x \to n^+} (x) - \lim\limits_{x \to n^+} [x] = n - n = 0$

It is observed that the left and right hand limits of f at $x = n$ do not coincide.

Therefore, f is not continuous at $x = n$

Hence, g is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = p$?

Answer

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at $x = p$

At $x = \pi$, $f(x) = f(\pi) = \pi^2 - \sin\pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Consider $\lim\limits_{x\to\pi} f(x) = \lim\limits_{x\to\pi}\left(x^2 - \sin x + 5\right)$

Put $x = \pi + h$

If $x \to \pi$, then it is evident that $h \to 0$

$\therefore \lim\limits_{x\to\pi} f(x) = \lim\limits_{x\to\pi}\left(x^2 - \sin x + 5\right)$

$\qquad = \lim\limits_{h\to0}\left[(\pi+h)^2 - \sin(\pi+h) + 5\right]$

$\qquad = \lim\limits_{h\to0}(\pi+h)^2 - \lim\limits_{h\to0}\sin(\pi+h) + \lim\limits_{h\to0}5$

$\qquad = (\pi+0)^2 - \lim\limits_{h\to0}\left[\sin\pi\cos h + \cos\pi\sin h\right] + 5$

$\qquad = \pi^2 - \lim\limits_{h\to0}\sin\pi\cos h - \lim\limits_{h\to0}\cos\pi\sin h + 5$

$\qquad = \pi^2 - \sin\pi\cos 0 - \cos\pi\sin 0 + 5$

$\qquad = \pi^2 - 0\times1 - (-1)\times0 + 5$

$\qquad = \pi^2 + 5$

$\therefore \lim\limits_{x\to\pi} f(x) = f(\pi)$

Therefore, the given function f is continuous at $x = \pi$

Question 21:

Discuss the continuity of the following functions.

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \times \cos x$

Answer

It is known that if g and h are two continuous functions, then

$g + h,\ g - h,\ \text{and}\ g.h$ are also continuous.

It has to proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$g(c) = \sin c$

$$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x$$

$$= \lim_{h \to 0} \sin(c+h)$$

$$= \lim_{h \to 0}\left[\sin c \cos h + \cos c \sin h\right]$$

$$= \lim_{h \to 0}(\sin c \cos h) + \lim_{h \to 0}(\cos c \sin h)$$

$$= \sin c \cos 0 + \cos c \sin 0$$

$$= \sin c + 0$$

$$= \sin c$$

$\therefore \quad \lim_{x \to c} g(x) = g(c)$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$$

$$= \lim_{h \to 0} \cos(c+h)$$

$$= \lim_{h \to 0}\left[\cos c \cos h - \sin c \sin h\right]$$

$$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$$

$$= \cos c \cos 0 - \sin c \sin 0$$

$$= \cos c \times 1 - \sin c \times 0$$

$$= \cos c$$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, h is a continuous function.

Therefore, it can be concluded that

(a) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function

(b) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function

(c) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function

Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer

It is known that if g and h are two continuous functions, then

(i) $\dfrac{h(x)}{g(x)}$, $g(x) \neq 0$ is continuous

(ii) $\dfrac{1}{g(x)}$, $g(x) \neq 0$ is continuous

(iii) $\dfrac{1}{h(x)}$, $h(x) \neq 0$ is continuous

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$g(c) = \sin c$

$\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} \sin x$

$\qquad = \lim\limits_{h \to 0} \sin(c + h)$

$\qquad = \lim\limits_{h \to 0} \left[\sin c \cos h + \cos c \sin h \right]$

$\qquad = \lim\limits_{h \to 0} \left(\sin c \cos h \right) + \lim\limits_{h \to 0} \left(\cos c \sin h \right)$

$\qquad = \sin c \cos 0 + \cos c \sin 0$

$\qquad = \sin c + 0$

$\qquad = \sin c$

$\therefore \ \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$$

$$= \lim_{h \to 0} \cos(c+h)$$

$$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$$

$$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$$

$$= \cos c \cos 0 - \sin c \sin 0$$

$$= \cos c \times 1 - \sin c \times 0$$

$$= \cos c$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is continuous function.

It can be concluded that,

$$\csc x = \frac{1}{\sin x}, \quad \sin x \neq 0 \text{ is continuous}$$

$$\Rightarrow \csc x, \ x \neq n\pi \ (n \in Z) \text{ is continuous}$$

Therefore, cosecant is continuous except at $x = np$, n Î **Z**

$$\sec x = \frac{1}{\cos x}, \quad \cos x \neq 0 \text{ is continuous}$$

$$\Rightarrow \sec x, \ x \neq (2n+1)\frac{\pi}{2} \ (n \in Z) \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2} \ (n \in Z)$

$$\cot x = \frac{\cos x}{\sin x}, \quad \sin x \neq 0 \text{ is continuous}$$

$$\Rightarrow \cot x, \ x \neq n\pi \ (n \in Z) \text{ is continuous}$$

Therefore, cotangent is continuous except at $x = np$, n Î **Z**

Question 23:

Find the points of discontinuity of f, where

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

Answer

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

It is evident that *f* is defined at all points of the real line.

Let *c* be a real number.

Case I:

If $c < 0$, then $f(c) = \dfrac{\sin c}{c}$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}\left(\dfrac{\sin x}{x}\right) = \dfrac{\sin c}{c}$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* < 0

Case II:

If $c > 0$, then $f(c) = c+1$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x+1) = c+1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 0

Case III:

If $c = 0$, then $f(c) = f(0) = 0 + 1 = 1$

The left hand limit of *f* at *x* = 0 is,

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

The right hand limit of *f* at *x* = 0 is,

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+}(x+1) = 1$

$\therefore \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0)$

Therefore, *f* is continuous at *x* = 0

From the above observations, it can be concluded that *f* is continuous at all points of the real line.

Thus, *f* has no point of discontinuity.

Question 24:

Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

Answer

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = c^2 \sin \dfrac{1}{c}$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(x^2 \sin \dfrac{1}{x} \right) = \left(\lim_{x \to c} x^2 \right) \left(\lim_{x \to c} \sin \dfrac{1}{x} \right) = c^2 \sin \dfrac{1}{c}$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points $x \neq 0$

Case II:

If $c = 0$, then $f(0) = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \le \sin\frac{1}{x} \le 1,\ x \ne 0$

$$\Rightarrow -x^2 \le \sin\frac{1}{x} \le x^2$$

$$\Rightarrow \lim_{x \to 0}\left(-x^2\right) \le \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x \to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$$

$$\Rightarrow \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x \to 0^-} f(x) = 0$$

Similarly, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$$\therefore \lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x)$$

Therefore, f is continuous at x = 0

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 25:

Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Answer

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = \sin c - \cos c$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (\sin x - \cos x) = \sin c - \cos c$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* ≠ 0

Case II:

If $c = 0$, then $f(0) = -1$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\therefore \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0)$

Therefore, *f* is continuous at *x* = 0

From the above observations, it can be concluded that *f* is continuous at every point of the real line.

Thus, *f* is a continuous function.

Question 26:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases} \qquad \text{at } x = \dfrac{\pi}{2}$$

Answer

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

The given function *f* is continuous at $x = \dfrac{\pi}{2}$, if *f* is defined at $x = \dfrac{\pi}{2}$ and if the value of the *f*

at $x = \dfrac{\pi}{2}$ equals the limit of *f* at $x = \dfrac{\pi}{2}$.

It is evident that f is defined at $x = \dfrac{\pi}{2}$ and $f\left(\dfrac{\pi}{2}\right) = 3$

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \dfrac{k \cos x}{\pi - 2x}$$

Put $x = \dfrac{\pi}{2} + h$

Then, $x \to \dfrac{\pi}{2} \Rightarrow h \to 0$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \dfrac{k \cos x}{\pi - 2x} = \lim_{h \to 0} \dfrac{k \cos\left(\dfrac{\pi}{2} + h\right)}{\pi - 2\left(\dfrac{\pi}{2} + h\right)}$$

$$= k \lim_{h \to 0} \dfrac{-\sin h}{-2h} = \dfrac{k}{2} \lim_{h \to 0} \dfrac{\sin h}{h} = \dfrac{k}{2} . 1 = \dfrac{k}{2}$$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\dfrac{\pi}{2}\right)$$

$$\Rightarrow \dfrac{k}{2} = 3$$

$$\Rightarrow k = 6$$

Therefore, the required value of k is 6.

Question 27:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases} \qquad \text{at } x = 2$$

Answer

The given function is $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$

The given function f is continuous at $x = 2$, if f is defined at $x = 2$ and if the value of f at $x = 2$ equals the limit of f at $x = 2$

It is evident that f is defined at $x = 2$ and $f(2) = k(2)^2 = 4k$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \to 2^-} (kx^2) = \lim_{x \to 2^+} (3) = 4k$$

$$\Rightarrow k \times 2^2 = 3 = 4k$$

$$\Rightarrow 4k = 3 = 4k$$

$$\Rightarrow 4k = 3$$

$$\Rightarrow k = \frac{3}{4}$$

Therefore, the required value of k is $\dfrac{3}{4}$.

Question 28:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases} \qquad \text{at } x = \pi$$

Answer

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function f is continuous at x = p, if f is defined at x = p and if the value of f at x = p equals the limit of f at x = p

It is evident that f is defined at x = p and $f(\pi) = k\pi + 1$

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x \to \pi^-} (kx+1) = \lim_{x \to \pi^+} \cos x = k\pi + 1$$

$$\Rightarrow k\pi + 1 = \cos \pi = k\pi + 1$$

$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the required value of k is $-\dfrac{2}{\pi}$.

Question 29:

Find the values of k so that the function f is continuous at the indicated point.

$f(x) = \begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$ at $x = 5$

Answer

$f(x) = \begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$

The given function f is continuous at $x = 5$, if f is defined at $x = 5$ and if the value of f at $x = 5$ equals the limit of f at $x = 5$

It is evident that f is defined at $x = 5$ and $f(5) = kx+1 = 5k+1$

$\lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^+} f(x) = f(5)$

$\Rightarrow \lim\limits_{x \to 5^-} (kx+1) = \lim\limits_{x \to 5^+} (3x-5) = 5k+1$

$\Rightarrow 5k+1 = 15-5 = 5k+1$

$\Rightarrow 5k+1 = 10$

$\Rightarrow 5k = 9$

$\Rightarrow k = \dfrac{9}{5}$

Therefore, the required value of k is $\dfrac{9}{5}$.

Question 30:

Find the values of a and b such that the function defined by

$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$

is a continuous function.

Answer

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$

It is evident that the given function f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at $x = 2$ and $x = 10$

Since f is continuous at $x = 2$, we obtain

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$

$\Rightarrow \lim\limits_{x \to 2^-} (5) = \lim\limits_{x \to 2^+} (ax+b) = 5$

$\Rightarrow 5 = 2a+b = 5$

$\Rightarrow 2a+b = 5$ ...(1)

Since f is continuous at $x = 10$, we obtain

$\lim\limits_{x \to 10^-} f(x) = \lim\limits_{x \to 10^+} f(x) = f(10)$

$\Rightarrow \lim\limits_{x \to 10^-} (ax+b) = \lim\limits_{x \to 10^+} (21) = 21$

$\Rightarrow 10a+b = 21 = 21$

$\Rightarrow 10a+b = 21$ ...(2)

On subtracting equation (1) from equation (2), we obtain

8a = 16

$\Rightarrow a = 2$

By putting $a = 2$ in equation (1), we obtain

2 × 2 + b = 5

$\Rightarrow$ 4 + b = 5

$\Rightarrow b = 1$

Therefore, the values of *a* and *b* for which *f* is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Answer

The given function is $f(x) = \cos(x^2)$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = \cos x$ and $h(x) = x^2$

$$\left[\because (goh)(x) = g\big(h(x)\big) = g\big(x^2\big) = \cos\big(x^2\big) = f(x) \right]$$

It has to be first proved that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that *g* is defined for every real number.

Let *c* be a real number.

Then, $g(c) = \cos c$

Put $x = c + h$

If $x \to c$, then $h \to 0$

$$\lim_{x \to c} g(x) = \lim_{x \to c} \cos x$$

$$= \lim_{h \to 0} \cos(c + h)$$

$$= \lim_{h \to 0} \left[\cos c \cos h - \sin c \sin h \right]$$

$$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$$

$$= \cos c \cos 0 - \sin c \sin 0$$

$$= \cos c \times 1 - \sin c \times 0$$

$$= \cos c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, $g(x) = \cos x$ is continuous function.

$h(x) = x^2$

Clearly, h is defined for every real number.

Let k be a real number, then $h(k) = k^2$

$$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$$

$$\therefore \lim_{x \to k} h(x) = h(k)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h, such that $(g \circ h)$ is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Question 32:

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Answer

The given function is $f(x) = |\cos x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = |x|$ and $h(x) = \cos x$

$$\left[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x) \right]$$

It has to be first proved that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} x = c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} (-x) = 0$

$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} (x) = 0$

$\therefore \lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^+} (x) = g(0)$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$\begin{aligned}
\lim\limits_{x \to c} h(x) &= \lim\limits_{x \to c} \cos x \\
&= \lim\limits_{h \to 0} \cos(c + h) \\
&= \lim\limits_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim\limits_{h \to 0} \cos c \cos h - \lim\limits_{h \to 0} \sin c \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c
\end{aligned}$

$\therefore \lim\limits_{x \to c} h(x) = h(c)$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions g and h, such that $(g \, o \, h)$ is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then $(f \, o \, g)$ is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function.

Question 33:

Examine that $\sin|x|$ is a continuous function.

Answer

Let $f(x) = \sin|x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

$f = g$ o h, where $g(x) = |x|$ and $h(x) = \sin x$

$\left[\because (goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x) \right]$

It has to be proved first that $g(x) = |x|$ and $h(x) = \sin x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} (-x) = -c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, *g* is continuous at all points *x*, such that *x* < 0

Case II:

If $c > 0$, then $g(c) = c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} x = c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, *g* is continuous at all points *x*, such that *x* > 0

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$

$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$$

Therefore, *g* is continuous at *x* = 0

From the above three observations, it can be concluded that *g* is continuous at all points.

h (*x*) = sin *x*

It is evident that *h* (*x*) = sin *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *k*

If *x* → *c*, then *k* → 0

h (*c*) = sin *c*

$$h(c) = \sin c$$

$$\begin{aligned}\lim_{x \to c} h(x) &= \lim_{x \to c} \sin x \\ &= \lim_{k \to 0} \sin (c + k) \\ &= \lim_{k \to 0} \left[\sin c \cos k + \cos c \sin k \right] \\ &= \lim_{k \to 0} (\sin c \cos k) + \lim_{h \to 0} (\cos c \sin k) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c + 0 \\ &= \sin c \end{aligned}$$

$$\therefore \lim_{x \to c} h(x) = g(c)$$

Therefore, *h* is a continuous function.

It is known that for real valued functions *g* and *h*, such that (*g* o *h*) is defined at *c*, if *g* is continuous at *c* and if *f* is continuous at *g* (*c*), then (*f* o *g*) is continuous at *c*.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of *f* defined by $f(x) = |x| - |x + 1|$.

Answer

The given function is $f(x) = |x| - |x + 1|$

The two functions, g and h, are defined as

$g(x) = |x|$ and $h(x) = |x+1|$

Then, $f = g - h$

The continuity of g and h is examined first.

$g(x) = |x|$ can be written as

$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c}(-x) = -c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} x = c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-}(-x) = 0$

$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+}(x) = 0$

$\therefore \lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^+}(x) = g(0)$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = |x+1|$ can be written as

$h(x) = \begin{cases} -(x+1), & \text{if, } x < -1 \\ x+1, & \text{if } x \geq -1 \end{cases}$

Clearly, h is defined for every real number.

Let c be a real number.

Case I:

If $c < -1$, then $h(c) = -(c+1)$ and $\lim\limits_{x \to c} h(x) = \lim\limits_{x \to c} \left[-(x+1) \right] = -(c+1)$

$\therefore \lim\limits_{x \to c} h(x) = h(c)$

Therefore, h is continuous at all points x, such that $x < -1$

Case II:

If $c > -1$, then $h(c) = c+1$ and $\lim\limits_{x \to c} h(x) = \lim\limits_{x \to c} (x+1) = c+1$

$\therefore \lim\limits_{x \to c} h(x) = h(c)$

Therefore, h is continuous at all points x, such that $x > -1$

Case III:

If $c = -1$, then $h(c) = h(-1) = -1+1 = 0$

$\lim\limits_{x \to -1^-} h(x) = \lim\limits_{x \to -1^-} \left[-(x+1) \right] = -(-1+1) = 0$

$\lim\limits_{x \to -1^+} h(x) = \lim\limits_{x \to -1^+} (x+1) = (-1+1) = 0$

$\therefore \lim\limits_{x \to -1^-} h(x) = \lim\limits_{h \to -1^+} h(x) = h(-1)$

Therefore, h is continuous at $x = -1$

From the above three observations, it can be concluded that h is continuous at all points of the real line.

g and h are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, f has no point of discontinuity.

Exercise 5.2

Question 1:

Differentiate the functions with respect to *x*.

$$\sin\left(x^2 + 5\right)$$

Answer

$$\text{Let } f(x) = \sin\left(x^2 + 5\right), \; u(x) = x^2 + 5, \text{ and } v(t) = \sin t$$

$$\text{Then, } (vou)(x) = v(u(x)) = v(x^2 + 5) = \tan(x^2 + 5) = f(x)$$

Thus, *f* is a composite of two functions.

$$\text{Put } t = u(x) = x^2 + 5$$

Then, we obtain

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos\left(x^2 + 5\right)$$

$$\frac{dt}{dx} = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5) = 2x + 0 = 2x$$

$$\text{Therefore, by chain rule, } \frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos\left(x^2 + 5\right) \times 2x = 2x\cos\left(x^2 + 5\right)$$

Alternate method

$$\frac{d}{dx}\left[\sin\left(x^2 + 5\right)\right] = \cos\left(x^2 + 5\right) \cdot \frac{d}{dx}\left(x^2 + 5\right)$$

$$= \cos\left(x^2 + 5\right) \cdot \left[\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(5)\right]$$

$$= \cos\left(x^2 + 5\right) \cdot \left[2x + 0\right]$$

$$= 2x\cos\left(x^2 + 5\right)$$

Question 2:

Differentiate the functions with respect to *x*.

$$\cos\left(\sin x\right)$$

Answer

Let $f(x) = \cos(\sin x), u(x) = \sin x$, and $v(t) = \cos t$

Then, $(vou)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x)$

Thus, f is a composite function of two functions.

Put $t = u(x) = \sin x$

$\therefore \dfrac{dv}{dt} = \dfrac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)$

$\dfrac{dt}{dx} = \dfrac{d}{dx}(\sin x) = \cos x$

By chain rule, $\dfrac{df}{dx} = \dfrac{dv}{dt} \cdot \dfrac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$

Alternate method

$\dfrac{d}{dx}[\cos(\sin x)] = -\sin(\sin x) \cdot \dfrac{d}{dx}(\sin x) = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$

Question 3:

Differentiate the functions with respect to x.

$\sin(ax + b)$

Answer

Let $f(x) = \sin(ax + b), u(x) = ax + b$, and $v(t) = \sin t$

Then, $(vou)(x) = v(u(x)) = v(ax + b) = \sin(ax + b) = f(x)$

Thus, f is a composite function of two functions, u and v.

Put $t = u(x) = ax + b$

Therefore,

$\dfrac{dv}{dt} = \dfrac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$

$\dfrac{dt}{dx} = \dfrac{d}{dx}(ax + b) = \dfrac{d}{dx}(ax) + \dfrac{d}{dx}(b) = a + 0 = a$

Hence, by chain rule, we obtain

$\dfrac{df}{dx} = \dfrac{dv}{dt} \cdot \dfrac{dt}{dx} = \cos(ax + b) \cdot a = a\cos(ax + b)$

Alternate method

$$\frac{d}{dx}\left[\sin\left(ax+b\right)\right]=\cos\left(ax+b\right).\frac{d}{dx}\left(ax+b\right)$$

$$=\cos\left(ax+b\right)\cdot\left[\frac{d}{dx}\left(ax\right)+\frac{d}{dx}\left(b\right)\right]$$

$$=\cos\left(ax+b\right).\left(a+0\right)$$

$$=a\cos\left(ax+b\right)$$

Question 4:

Differentiate the functions with respect to *x*.

$$\sec\left(\tan\left(\sqrt{x}\right)\right)$$

Answer

Let $f\left(x\right)=\sec\left(\tan\sqrt{x}\right),u\left(x\right)=\sqrt{x},v\left(t\right)=\tan t,$ and $w\left(s\right)=\sec s$

Then, $\left(wovou\right)\left(x\right)=w\left[v\left(u\left(x\right)\right)\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f\left(x\right)$

Thus, *f* is a composite function of three functions, *u, v,* and *w*.

Put $s=v\left(t\right)=\tan t$ and $t=u\left(x\right)=\sqrt{x}$

Then, $\dfrac{dw}{ds}=\dfrac{d}{ds}\left(\sec s\right)=\sec s\tan s=\sec\left(\tan t\right).\tan\left(\tan t\right)$ $\qquad\left[s=\tan t\right]$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\qquad\qquad\left[t=\sqrt{x}\right]$$

$\dfrac{ds}{dt}=\dfrac{d}{dt}\left(\tan t\right)=\sec^{2}t=\sec^{2}\sqrt{x}$

$\dfrac{dt}{dx}=\dfrac{d}{dx}\left(\sqrt{x}\right)=\dfrac{d}{dx}\left(x^{\frac{1}{2}}\right)=\dfrac{1}{2}\cdot x^{\frac{1}{2}-1}=\dfrac{1}{2\sqrt{x}}$

Hence, by chain rule, we obtain

$$\frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

$$= \sec\left(\tan \sqrt{x}\right) \cdot \tan\left(\tan \sqrt{x}\right) \times \sec^2 \sqrt{x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} \sec^2 \sqrt{x} \sec\left(\tan \sqrt{x}\right) \tan\left(\tan \sqrt{x}\right)$$

$$= \frac{\sec^2 \sqrt{x} \sec\left(\tan \sqrt{x}\right) \tan\left(\tan \sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan \sqrt{x}\right)\right] = \sec\left(\tan \sqrt{x}\right) \cdot \tan\left(\tan \sqrt{x}\right) \cdot \frac{d}{dx}\left(\tan \sqrt{x}\right)$$

$$= \sec\left(\tan \sqrt{x}\right) \cdot \tan\left(\tan \sqrt{x}\right) \cdot \sec^2\left(\sqrt{x}\right) \cdot \frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \sec\left(\tan \sqrt{x}\right) \cdot \tan\left(\tan \sqrt{x}\right) \cdot \sec^2\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\sec\left(\tan \sqrt{x}\right) \cdot \tan\left(\tan \sqrt{x}\right) \sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5:

Differentiate the functions with respect to *x*.

$$\frac{\sin(ax+b)}{\cos(cx+d)}$$

Answer

The given function is $f(x) = \frac{\sin(ax+b)}{\cos(cx+d)} = \frac{g(x)}{h(x)}$, where *g* (*x*) = sin (*ax* + *b*) and

h (*x*) = cos (*cx* + *d*)

$$\therefore f' = \frac{g'h - gh'}{h^2}$$

Consider $g(x) = \sin(ax+b)$

Let $u(x) = ax+b, v(t) = \sin t$

Then, $(v \circ u)(x) = v(u(x)) = v(ax+b) = \sin(ax+b) = g(x)$

∴ *g* is a composite function of two functions, *u* and *v*.

Put $t = u(x) = ax + b$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Therefore, by chain rule, we obtain

$$g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b) \cdot a = a\cos(ax + b)$$

Consider $h(x) = \cos(cx + d)$

Let $p(x) = cx + d,\ q(y) = \cos y$

Then, $(qop)(x) = q(p(x)) = q(cx + d) = \cos(cx + d) = h(x)$

∴*h* is a composite function of two functions, *p* and *q*.

Put *y* = *p* (*x*) = *cx* + *d*

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx + d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx + d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Therefore, by chain rule, we obtain

$$h' = \frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} = -\sin(cx + d) \times c = -c\sin(cx + d)$$

$$\therefore f' = \frac{a\cos(ax+b)\cdot\cos(cx+d)-\sin(ax+b)\{-c\sin(cx+d)\}}{\left[\cos(cx+d)\right]^2}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)}+c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)}\times\frac{1}{\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d)+c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Question 6:

Differentiate the functions with respect to *x*.

$$\cos x^3.\sin^2\left(x^5\right)$$

Answer

The given function is $\cos x^3.\sin^2\left(x^5\right)$

$$\frac{d}{dx}\left[\cos x^3 \cdot \sin^2\left(x^5\right)\right] = \sin^2\left(x^5\right)\times\frac{d}{dx}\left(\cos x^3\right)+\cos x^3\times\frac{d}{dx}\left[\sin^2\left(x^5\right)\right]$$

$$= \sin^2\left(x^5\right)\times\left(-\sin x^3\right)\times\frac{d}{dx}\left(x^3\right)+\cos x^3\times2\sin\left(x^5\right)\cdot\frac{d}{dx}\left[\sin x^5\right]$$

$$= -\sin x^3\sin^2\left(x^5\right)\times3x^2+2\sin x^5\cos x^3\cdot\cos x^5\times\frac{d}{dx}\left(x^5\right)$$

$$= -3x^2\sin x^3\cdot\sin^2\left(x^5\right)+2\sin x^5\cos x^5\cos x^3\cdot\times5x^4$$

$$= 10x^4\sin x^5\cos x^5\cos x^3-3x^2\sin x^3\sin^2\left(x^5\right)$$

Question 7:

Differentiate the functions with respect to *x*.

$$2\sqrt{\cot\left(x^2\right)}$$

Answer

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right]$$

$$=2\cdot\frac{1}{2\sqrt{\cot\left(x^2\right)}}\times\frac{d}{dx}\left[\cot\left(x^2\right)\right]$$

$$=\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times-\text{cosec}^2\left(x^2\right)\times\frac{d}{dx}\left(x^2\right)$$

$$=-\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times\frac{1}{\sin^2\left(x^2\right)}\times\left(2x\right)$$

$$=\frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\sin x^2}$$

$$=\frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2\cos x^2}\sin x^2}$$

$$=\frac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$$

Question 8:

Differentiate the functions with respect to *x*.

$$\cos\left(\sqrt{x}\right)$$

Answer

$$\text{Let } f\left(x\right)=\cos\left(\sqrt{x}\right)$$

$$\text{Also, let } u\left(x\right)=\sqrt{x}$$

$$\text{And, } v\left(t\right)=\cos t$$

$$\text{Then, } \left(vou\right)\left(x\right)=v\big(u\left(x\right)\big)$$

$$=v\left(\sqrt{x}\right)$$

$$=\cos\sqrt{x}$$

$$=f\left(x\right)$$

Clearly, *f* is a composite function of two functions, *u* and *v*, such that

$$t=u\left(x\right)=\sqrt{x}$$

Then, $\dfrac{dt}{dx} = \dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{d}{dx}\left(x^{\frac{1}{2}}\right) = \dfrac{1}{2}x^{-\frac{1}{2}}$

$$= \dfrac{1}{2\sqrt{x}}$$

And, $\dfrac{dv}{dt} = \dfrac{d}{dt}\left(\cos t\right) = -\sin t$

$$= -\sin\left(\sqrt{x}\right)$$

By using chain rule, we obtain

$$\dfrac{dt}{dx} = \dfrac{dv}{dt}\cdot\dfrac{dt}{dx}$$

$$= -\sin\left(\sqrt{x}\right)\cdot\dfrac{1}{2\sqrt{x}}$$

$$= -\dfrac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$$

$$= -\dfrac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\dfrac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right] = -\sin\left(\sqrt{x}\right)\cdot\dfrac{d}{dx}\left(\sqrt{x}\right)$$

$$= -\sin\left(\sqrt{x}\right)\times\dfrac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= -\sin\sqrt{x}\times\dfrac{1}{2}x^{-\frac{1}{2}}$$

$$= \dfrac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question 9:

Prove that the function *f* given by

$f(x) = |x-1|$, $x \in \mathbf{R}$ is notdifferentiable at $x = 1$.

Answer

The given function is $f(x) = |x-1|$, $x \in \mathbf{R}$

It is known that a function f is differentiable at a point $x = c$ in its domain if both

$\lim\limits_{h\to 0^-} \dfrac{f(c+h)-f(c)}{h}$ and $\lim\limits_{h\to 0^+} \dfrac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x = 1$,

consider the left hand limit of f at $x = 1$

$\lim\limits_{h\to 0^-} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0^-} \dfrac{|1+h-1|-|1-1|}{h}$

$= \lim\limits_{h\to 0^-} \dfrac{|h|-0}{h} = \lim\limits_{h\to 0^-} \dfrac{-h}{h}$ $\left(h<0 \Rightarrow |h|=-h\right)$

$= -1$

Consider the right hand limit of f at $x = 1$

$\lim\limits_{h\to 0^+} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0^+} \dfrac{|1+h-1|-|1-1|}{h}$

$= \lim\limits_{h\to 0^+} \dfrac{|h|-0}{h} = \lim\limits_{h\to 0^+} \dfrac{h}{h}$ $\left(h>0 \Rightarrow |h|=h\right)$

$= 1$

Since the left and right hand limits of f at $x = 1$ are not equal, f is not differentiable at $x = 1$

Question 10:

Prove that the greatest integer function defined by $f(x)=[x], 0<x<3$ is not differentiable at $x = 1$ and $x = 2$.

Answer

The given function f is $f(x)=[x], 0<x<3$

It is known that a function f is differentiable at a point $x = c$ in its domain if both

$\lim\limits_{h\to 0^-} \dfrac{f(c+h)-f(c)}{h}$ and $\lim\limits_{h\to 0^+} \dfrac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x = 1$, consider the left hand limit of f at $x = 1$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[1+h] - [1]}{h}$$

$$= \lim_{h \to 0^-} \frac{0-1}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

Consider the right hand limit of f at $x = 1$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[1+h] - [1]}{h}$$

$$= \lim_{h \to 0^+} \frac{1-1}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = 1$ are not equal, f is not differentiable at

$x = 1$

To check the differentiability of the given function at $x = 2$, consider the left hand limit

of f at $x = 2$

$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \to 0^-} \frac{1-2}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

Consider the right hand limit of f at $x = 1$

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \to 0^+} \frac{2-2}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = 2$ are not equal, f is not differentiable at x

$= 2$

Question 1:

Find $\dfrac{dy}{dx}$:

$2x + 3y = \sin x$

Answer

The given relationship is $2x + 3y = \sin x$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(2x+3y) = \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos x$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x$$

$$\Rightarrow 3\frac{dy}{dx} = \cos x - 2$$

$$\therefore \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

Question 2:

Find $\dfrac{dy}{dx}$

$2x + 3y = \sin y$

Answer

The given relationship is $2x + 3y = \sin y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(\sin y)$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos y \frac{dy}{dx} \qquad \left[\text{By using chain rule}\right]$$

$$\Rightarrow 2 = (\cos y - 3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

Question 3:

Find $\frac{dy}{dx}$

$$ax + by^2 = \cos y$$

Answer

The given relationship is $ax + by^2 = \cos y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y)$$

$$\Rightarrow a + b\frac{d}{dx}(y^2) = \frac{d}{dx}(\cos y) \qquad \dots(1)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y) = -\sin y\frac{dy}{dx}$ $\qquad \dots(2)$

From (1) and (2), we obtain

$$a + b \times 2y\frac{dy}{dx} = -\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by + \sin y)\frac{dy}{dx} = -a$$

$$\therefore \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

Question 4:

Find $\frac{dy}{dx}$

$$xy + y^2 = \tan x + y$$

Answer

The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(\tan x + y)$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$\Rightarrow \left[y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} \right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow y \cdot 1 + x \cdot \frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x + 2y - 1)}$$

Question 5:

Find $\dfrac{dy}{dx}$

$$x^2 + xy + y^2 = 100$$

Answer

The given relationship is $x^2 + xy + y^2 = 100$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$\Rightarrow 2x + \left[y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} \right] + 2y\frac{dy}{dx} = 0 \qquad \left[\text{Using product rule and chain rule}\right]$$

$$\Rightarrow 2x + y \cdot 1 + x \cdot \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y + (x + 2y)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Question 6:

Find $\frac{dy}{dx}$

$x^3 + x^2y + xy^2 + y^3 = 81$

Answer

The given relationship is $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(x^3 + x^2y + xy^2 + y^3\right) = \frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^2y\right) + \frac{d}{dx}\left(xy^2\right) + \frac{d}{dx}\left(y^3\right) = 0$$

$$\Rightarrow 3x^2 + \left[y\frac{d}{dx}\left(x^2\right) + x^2\frac{dy}{dx} \right] + \left[y^2\frac{d}{dx}(x) + x\frac{d}{dx}\left(y^2\right) \right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + \left[y \cdot 2x + x^2\frac{dy}{dx} \right] + \left[y^2 \cdot 1 + x \cdot 2y \cdot \frac{dy}{dx} \right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \left(x^2 + 2xy + 3y^2\right)\frac{dy}{dx} + \left(3x^2 + 2xy + y^2\right) = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\left(3x^2 + 2xy + y^2\right)}{\left(x^2 + 2xy + 3y^2\right)}$$

Question 7:

Find $\frac{dy}{dx}$

$\sin^2 y + \cos xy = \pi$

Answer

The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}\left(\pi\right)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right) + \frac{d}{dx}\left(\cos xy\right) = 0 \qquad\qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right) = 2\sin y \frac{d}{dx}\left(\sin y\right) = 2\sin y \cos y \frac{dy}{dx} \qquad\qquad ...(2)$$

$$\frac{d}{dx}\left(\cos xy\right) = -\sin xy \frac{d}{dx}\left(xy\right) = -\sin xy\left[y\frac{d}{dx}\left(x\right) + x\frac{dy}{dx}\right]$$

$$= -\sin xy\left[y.1 + x\frac{dy}{dx}\right] = -y\sin xy - x\sin xy \frac{dy}{dx} \qquad\qquad ...(3)$$

From (1), (2), and (3), we obtain

$$2\sin y \cos y \frac{dy}{dx} - y\sin xy - x\sin xy \frac{dy}{dx} = 0$$

$$\Rightarrow \left(2\sin y \cos y - x\sin xy\right)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \left(\sin 2y - x\sin xy\right)\frac{dy}{dx} = y\sin xy$$

$$\therefore \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Question 8:

Find $\dfrac{dy}{dx}$

$\sin^2 x + \cos^2 y = 1$

Answer

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 x + \cos^2 y\right) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right) + \frac{d}{dx}\left(\cos^2 y\right) = 0$$

$$\Rightarrow 2\sin x \cdot \frac{d}{dx}\left(\sin x\right) + 2\cos y \cdot \frac{d}{dx}\left(\cos y\right) = 0$$

$$\Rightarrow 2\sin x \cos x + 2\cos y\left(-\sin y\right) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x - \sin 2y \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

Question 9:

Find $\dfrac{dy}{dx}$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Answer

The given relationship is $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{2x}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin y\right) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y \frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad \qquad ...(1)$$

The function, $\dfrac{2x}{1+x^2}$, is of the form of $\dfrac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) = \frac{(1+x^2)\cdot\frac{d}{dx}(2x) - 2x\cdot\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\cdot2 - 2x\cdot[0+2x]}{(1+x^2)^2} = \frac{2+2x^2-4x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2} \qquad ...(2)$$

Also, $\sin y = \dfrac{2x}{1+x^2}$

$$\Rightarrow \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{2x}{1+x^2}\right)^2} = \sqrt{\frac{(1+x^2)^2-4x^2}{(1+x^2)^2}}$$

$$= \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}} = \frac{1-x^2}{1+x^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2}\times\frac{dy}{dx} = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

Question 10:

Find $\dfrac{dy}{dx}$

$$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

Answer

The given relationship is $y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

$$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \tan y = \frac{3x - x^3}{1 - 3x^2} \qquad\qquad ...(1)$$

It is known that, $\tan y = \dfrac{3\tan\frac{y}{3} - \tan^3\frac{y}{3}}{1 - 3\tan^2\frac{y}{3}}$ $\qquad\qquad ...(2)$

Comparing equations (1) and (2), we obtain

$$x = \tan\frac{y}{3}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3} \cdot \frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3} \cdot \frac{1}{3} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{\sec^2\frac{y}{3}} = \frac{3}{1 + \tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{1 + x^2}$$

Question 11:

Find $\dfrac{dy}{dx}$

$$y = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right), 0 < x < 1$$

Answer

The given relationship is,

$$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1 - \tan^2 \frac{y}{2}}{1 + \tan^2 \frac{y}{2}} = \frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan\frac{y}{2} = x$$

Differentiating this relationship with respect to x, we obtain

$$\sec^2 \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec^2 \frac{y}{2} \times \frac{1}{2}\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2 \frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1 + \tan^2 \frac{y}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

Question 12:

Find $\dfrac{dy}{dx}$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), \quad 0 < x < 1$$

Answer

The given relationship is $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad \text{...(1)}$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$= \sqrt{\frac{\left(1+x^2\right)^2 - \left(1-x^2\right)^2}{\left(1+x^2\right)^2}} = \sqrt{\frac{4x^2}{\left(1+x^2\right)^2}} = \frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y) = \frac{2x}{1+x^2}\frac{dy}{dx} \qquad \text{...(2)}$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\left(1+x^2\right)\cdot\left(1-x^2\right)' - \left(1-x^2\right)\cdot\left(1+x^2\right)'}{\left(1+x^2\right)^2} \qquad \text{[Using quotient rule]}$$

$$= \frac{\left(1+x^2\right)(-2x) - \left(1-x^2\right)\cdot(2x)}{\left(1+x^2\right)^2}$$

$$= \frac{-2x - 2x^3 - 2x + 2x^3}{\left(1+x^2\right)^2}$$

$$= \frac{-4x}{\left(1+x^2\right)^2} \qquad \text{...(3)}$$

From (1), (2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{-2}{1+x^2}$$

Alternate method

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow\sin y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow\left(1+x^2\right)\sin y=1-x^2$$

$$\Rightarrow\left(1+\sin y\right)x^2=1-\sin y$$

$$\Rightarrow x^2=\frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2=\frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x=\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x=\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x=\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left[\tan\left(\frac{\pi}{4} - \frac{y}{2}\right)\right]$$

$$\Rightarrow 1 = \sec^2\left(\frac{\pi}{4} - \frac{y}{2}\right) \cdot \frac{d}{dx}\left(\frac{\pi}{4} - \frac{y}{2}\right)$$

$$\Rightarrow 1 = \left[1 + \tan^2\left(\frac{\pi}{4} - \frac{y}{2}\right)\right] \cdot \left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow 1 = \left(1 + x^2\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Question 13:

Find $\dfrac{dy}{dx}$

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$$

Answer

The given relationship is $y = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y = \frac{2x}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\cos y) = \frac{d}{dx} \cdot \left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y \cdot \frac{dy}{dx} = \frac{\left(1+x^2\right) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx} = \frac{\left(1+x^2\right)\times 2 - 2x\cdot 2x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx} = -\left[\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}\right]$$

$$\Rightarrow \sqrt{\frac{\left(1+x^2\right)^2 - 4x^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}\cdot\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Question 14:

Find $\dfrac{dy}{dx}$

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right),\quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

Answer

The given relationship is $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin y = 2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\cos y \frac{dy}{dx} = 2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$\Rightarrow \sqrt{1-\sin^2 y}\,\frac{dy}{dx} = 2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right]$$

$$\Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\,\frac{dy}{dx} = 2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{1-4x^2\left(1-x^2\right)}\,\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{\left(1-2x^2\right)^2}\,\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \left(1-2x^2\right)\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Question 15:

Find $\dfrac{dy}{dx}$

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$$

Answer

The given relationship is $y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y = \frac{1}{2x^2 - 1}$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos \frac{y}{2}\right)$$

$$\Rightarrow 1 = -\sin \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

Exercise 5.4

Question 1:

Differentiate the following w.r.t. x:

$$\frac{e^x}{\sin x}$$

Answer

Let $y = \dfrac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sin x \dfrac{d}{dx}\left(e^x\right) - e^x \dfrac{d}{dx}\left(\sin x\right)}{\sin^2 x}$$

$$= \frac{\sin x.\left(e^x\right) - e^x \cdot \left(\cos x\right)}{\sin^2 x}$$

$$= \frac{e^x\left(\sin x - \cos x\right)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}$$

Question 2:

Differentiate the following w.r.t. x:

$$e^{\sin^{-1} x}$$

Answer

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1}x} \cdot \frac{d}{dx}\left(\sin^{-1}x\right)$$

$$= e^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}, x \in (-1,1)$$

Question 2:

Show that the function given by $f(x) = e^{2x}$ is strictly increasing on **R**.

Answer

Let x_1 and x_2 be any two numbers in **R**.

Then, we have:

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow e^{2x_1} < e^{2x_2} \Rightarrow f(x_1) < f(x_2)$$

Hence, *f* is strictly increasing on **R**.

Question 3:

Differentiate the following w.r.t. *x*:

$$e^{x^3}$$

Answer

Let $y = e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right) = e^{x^3} \cdot \frac{d}{dx}\left(x^3\right) = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

Question 4:

Differentiate the following w.r.t. *x*:

$$\sin\left(\tan^{-1}e^{-x}\right)$$

Answer

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{1}{1+\left(e^{-x}\right)^2} \cdot \frac{d}{dx}\left(e^{-x}\right)$$

$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \cdot e^{-x} \cdot \frac{d}{dx}(-x)$$

$$= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \times (-1)$$

$$= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiate the following w.r.t. *x*:

$$\log\left(\cos e^x\right)$$

Answer

Let $y = \log\left(\cos e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\cos e^x\right)\right]$$

$$= \frac{1}{\cos e^x} \cdot \frac{d}{dx}\left(\cos e^x\right)$$

$$= \frac{1}{\cos e^x} \cdot \left(-\sin e^x\right) \cdot \frac{d}{dx}\left(e^x\right)$$

$$= \frac{-\sin e^x}{\cos e^x} \cdot e^x$$

$$= -e^x \tan e^x, \, e^x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{N}$$

Question 6:

Differentiate the following w.r.t. x:

$$e^x + e^{x^2} + \ldots + e^{x^5}$$

Answer

$$\frac{d}{dx}\left(e^x + e^{x^2} + \ldots + e^{x^5}\right)$$

$$= \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$$

$$= e^x + \left[e^{x^2} \times \frac{d}{dx}\left(x^2\right)\right] + \left[e^{x^3} \cdot \frac{d}{dx}\left(x^3\right)\right] + \left[e^{x^4} \cdot \frac{d}{dx}\left(x^4\right)\right] + \left[e^{x^5} \cdot \frac{d}{dx}\left(x^5\right)\right]$$

$$= e^x + \left(e^{x^2} \times 2x\right) + \left(e^{x^3} \times 3x^2\right) + \left(e^{x^4} \times 4x^3\right) + \left(e^{x^5} \times 5x^4\right)$$

$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

Question 7:

Differentiate the following w.r.t. x:

$$\sqrt{e^{\sqrt{x}}}, x > 0$$

Answer

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

By differentiating this relationship with respect to x, we obtain

$y^2 = e^{\sqrt{x}}$

$\Rightarrow 2y\dfrac{dy}{dx} = e^{\sqrt{x}} \dfrac{d}{dx}\left(\sqrt{x}\right)$ [By applying the chain rule]

$\Rightarrow 2y\dfrac{dy}{dx} = e^{\sqrt{x}} \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{x}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{4y\sqrt{x}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x > 0$

Question 8:

Differentiate the following w.r.t. *x*:

$\log\left(\log x\right), x > 1$

Answer

Let $y = \log\left(\log x\right)$

By using the chain rule, we obtain

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left[\log\left(\log x\right)\right]$

$= \dfrac{1}{\log x} \cdot \dfrac{d}{dx}\left(\log x\right)$

$= \dfrac{1}{\log x} \cdot \dfrac{1}{x}$

$= \dfrac{1}{x \log x}, x > 1$

Question 9:

Differentiate the following w.r.t. *x*:

$\dfrac{\cos x}{\log x}, x > 0$

Answer

Let $y = \dfrac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\dfrac{d}{dx}(\cos x) \times \log x - \cos x \times \dfrac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{-\sin x \log x - \cos x \times \dfrac{1}{x}}{(\log x)^2}$$

$$= \frac{-[x \log x.\sin x + \cos x]}{x(\log x)^2}, x > 0$$

Question 10:

Differentiate the following w.r.t. *x*:

$\cos(\log x + e^x), x > 0$

Answer

Let $y = \cos(\log x + e^x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = -\sin(\log x + e^x) \cdot \frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x)\right]$$

$$= -\sin(\log x + e^x) \cdot \left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right)\sin(\log x + e^x), x > 0$$

Exercise 5.5

Question 1:

Differentiate the function with respect to *x*.

$\cos x . \cos 2x . \cos 3x$

Answer

Let $y = \cos x . \cos 2x . \cos 3x$

Taking logarithm on both the sides, we obtain

$\log y = \log(\cos x . \cos 2x . \cos 3x)$

$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$

Differentiating both sides with respect to *x*, we obtain

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{\cos x} \cdot \dfrac{d}{dx}(\cos x) + \dfrac{1}{\cos 2x} \cdot \dfrac{d}{dx}(\cos 2x) + \dfrac{1}{\cos 3x} \cdot \dfrac{d}{dx}(\cos 3x)$

$\Rightarrow \dfrac{dy}{dx} = y\left[-\dfrac{\sin x}{\cos x} - \dfrac{\sin 2x}{\cos 2x} \cdot \dfrac{d}{dx}(2x) - \dfrac{\sin 3x}{\cos 3x} \cdot \dfrac{d}{dx}(3x)\right]$

$\therefore \dfrac{dy}{dx} = -\cos x . \cos 2x . \cos 3x\left[\tan x + 2\tan 2x + 3\tan 3x\right]$

Question 2:

Differentiate the function with respect to *x*.

$\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Answer

Let $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking logarithm on both the sides, we obtain

$$\log y = \log \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y = \frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log\{(x-1)(x-2)\} - \log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\right]$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}\cdot\frac{d}{dx}(x-1) + \frac{1}{x-2}\cdot\frac{d}{dx}(x-2) - \frac{1}{x-3}\cdot\frac{d}{dx}(x-3) \\ -\frac{1}{x-4}\cdot\frac{d}{dx}(x-4) - \frac{1}{x-5}\cdot\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

Question 3:

Differentiate the function with respect to x.

$$\left(\log x\right)^{\cos x}$$

Answer

Let $y = \left(\log x\right)^{\cos x}$

Taking logarithm on both the sides, we obtain

$$\log y = \cos x \cdot \log\left(\log x\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\cos x) \times \log(\log x) + \cos x \times \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = -\sin x \log(\log x) + \cos x \times \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\sin x \log(\log x) + \frac{\cos x}{\log x} \times \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right]$$

Question 4:

Differentiate the function with respect to *x*.

$$x^x - 2^{\sin x}$$

Answer

Let $y = x^x - 2^{\sin x}$

Also, let $x^x = u$ and $2^{\sin x} = v$

$\therefore y = u - v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

u = *x*^{x}

Taking logarithm on both the sides, we obtain

$$\log u = x \log x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log x + x \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x(\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$$

v = 2$^{\sin x}$

Taking logarithm on both the sides with respect to *x*, we obtain

$\log v = \sin x \cdot \log 2$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{v} \cdot \dfrac{dv}{dx} = \log 2 \cdot \dfrac{d}{dx}(\sin x)$

$\Rightarrow \dfrac{dv}{dx} = v \log 2 \cos x$

$\Rightarrow \dfrac{dv}{dx} = 2^{\sin x} \cos x \log 2$

$\therefore \dfrac{dy}{dx} = x^x (1 + \log x) - 2^{\sin x} \cos x \log 2$

Question 5:

Differentiate the function with respect to x.

$(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

Answer

Let $y = (x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

Taking logarithm on both the sides, we obtain

$\log y = \log (x+3)^2 + \log (x+4)^3 + \log (x+5)^4$

$\Rightarrow \log y = 2 \log (x+3) + 3 \log (x+4) + 4 \log (x+5)$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{y} \cdot \dfrac{dy}{dx} = 2 \cdot \dfrac{1}{x+3} \cdot \dfrac{d}{dx}(x+3) + 3 \cdot \dfrac{1}{x+4} \cdot \dfrac{d}{dx}(x+4) + 4 \cdot \dfrac{1}{x+5} \cdot \dfrac{d}{dx}(x+5)$

$\Rightarrow \dfrac{dy}{dx} = y \left[\dfrac{2}{x+3} + \dfrac{3}{x+4} + \dfrac{4}{x+5} \right]$

$\Rightarrow \dfrac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\dfrac{2}{x+3} + \dfrac{3}{x+4} + \dfrac{4}{x+5} \right]$

$\Rightarrow \dfrac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\dfrac{2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)}{(x+3)(x+4)(x+5)} \right]$

$\Rightarrow \dfrac{dy}{dx} = (x+3)(x+4)^2 (x+5)^3 \cdot \left[2(x^2 + 9x + 20) + 3(x^2 + 8x + 15) + 4(x^2 + 7x + 12) \right]$

$\therefore \dfrac{dy}{dx} = (x+3)(x+4)^2 (x+5)^3 (9x^2 + 70x + 133)$

Question 6:

Differentiate the function with respect to *x*.

$$\left(x+\frac{1}{x} \right)^x + x^{\left(1+\frac{1}{x}\right)}$$

Answer

Let $y = \left(x+\dfrac{1}{x} \right)^x + x^{\left(1+\frac{1}{x}\right)}$

Also, let $u = \left(x+\dfrac{1}{x} \right)^x$ and $v = x^{\left(1+\frac{1}{x}\right)}$

$\therefore y = u + v$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(1)

Then, $u = \left(x+\dfrac{1}{x} \right)^x$

$\Rightarrow \log u = \log\left(x+\dfrac{1}{x} \right)^x$

$\Rightarrow \log u = x\log\left(x+\dfrac{1}{x} \right)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x) \times \log\left(x + \frac{1}{x}\right) + x \times \frac{d}{dx}\left[\log\left(x + \frac{1}{x}\right)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = 1 \times \log\left(x + \frac{1}{x}\right) + x \times \frac{1}{\left(x + \frac{1}{x}\right)} \cdot \frac{d}{dx}\left(x + \frac{1}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log\left(x + \frac{1}{x}\right) + \frac{x}{\left(x + \frac{1}{x}\right)} \times \left(1 - \frac{1}{x^2}\right)\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x\left[\log\left(x + \frac{1}{x}\right) + \frac{\left(x - \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x\left[\log\left(x + \frac{1}{x}\right) + \frac{x^2 - 1}{x^2 + 1}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x\left[\frac{x^2 - 1}{x^2 + 1} + \log\left(x + \frac{1}{x}\right)\right] \qquad \ldots(2)$$

$$v = x^{\left(1 + \frac{1}{x}\right)}$$

$$\Rightarrow \log v = \log\left[x^{\left(1 + \frac{1}{x}\right)}\right]$$

$$\Rightarrow \log v = \left(1 + \frac{1}{x}\right)\log x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \left[\frac{d}{dx} \left(1 + \frac{1}{x} \right) \right] \times \log x + \left(1 + \frac{1}{x} \right) \cdot \frac{d}{dx} \log x$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \left(-\frac{1}{x^2} \right) \log x + \left(1 + \frac{1}{x} \right) \cdot \frac{1}{x}$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = -\frac{\log x}{x^2} + \frac{1}{x} + \frac{1}{x^2}$$

$$\Rightarrow \frac{dv}{dx} = v \left[\frac{-\log x + x + 1}{x^2} \right]$$

$$\Rightarrow \frac{dv}{dx} = x^{\left(1 + \frac{1}{x} \right)} \left(\frac{x + 1 - \log x}{x^2} \right) \qquad \ldots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = \left(x + \frac{1}{x} \right)^x \left[\frac{x^2 - 1}{x^2 + 1} + \log \left(x + \frac{1}{x} \right) \right] + x^{\left(1 + \frac{1}{x} \right)} \left(\frac{x + 1 - \log x}{x^2} \right)$$

Question 7:

Differentiate the function with respect to *x*.

$$\left(\log x \right)^x + x^{\log x}$$

Answer

$$\text{Let } y = \left(\log x \right)^x + x^{\log x}$$

$$\text{Also, let } u = \left(\log x \right)^x \text{ and } v = x^{\log x}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(1)$$

$u = (\log x)^x$

$$\Rightarrow \log u = \log \left[\left(\log x \right)^x \right]$$

$$\Rightarrow \log u = x \log \left(\log x \right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \times \log(\log x) + x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log(\log x) + x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x\left[\log(\log x) + \frac{x}{\log x} \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x\left[\frac{\log(\log x) \cdot \log x + 1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{x-1}\left[1 + \log x. \log(\log x)\right] \qquad\qquad ...(2)$$

$$v = x^{\log x}$$

$$\Rightarrow \log v = \log\left(x^{\log x}\right)$$

$$\Rightarrow \log v = \log x \log x = (\log x)^2$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx}\left[(\log x)^2\right]$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = 2(\log x) \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dv}{dx} = 2v(\log x) \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x-1} \cdot \log x \qquad\qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\log x)^{x-1}\left[1 + \log x. \log(\log x)\right] + 2x^{\log x-1} \cdot \log x$$

Question 8:

Differentiate the function with respect to *x*.

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Answer

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(1)

$u = (\sin x)^x$

$\Rightarrow \log u = \log(\sin x)^x$

$\Rightarrow \log u = x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$\Rightarrow \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{d}{dx}(x) \times \log(\sin x) + x \times \dfrac{d}{dx}\big[\log(\sin x)\big]$

$\Rightarrow \dfrac{du}{dx} = u\left[1 \cdot \log(\sin x) + x \cdot \dfrac{1}{\sin x} \cdot \dfrac{d}{dx}(\sin x)\right]$

$\Rightarrow \dfrac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \dfrac{x}{\sin x} \cdot \cos x\right]$

$\Rightarrow \dfrac{du}{dx} = (\sin x)^x(x\cot x + \log\sin x)$ ...(2)

$v = \sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x, we obtain

$\dfrac{dv}{dx} = \dfrac{1}{\sqrt{1 - (\sqrt{x})^2}} \cdot \dfrac{d}{dx}(\sqrt{x})$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{1}{\sqrt{1-x}} \cdot \dfrac{1}{2\sqrt{x}}$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{1}{2\sqrt{x - x^2}}$ ...(3)

Therefore, from (1), (2), and (3), we obtain

$\dfrac{dy}{dx} = (\sin x)^x(x\cot x + \log\sin x) + \dfrac{1}{2\sqrt{x - x^2}}$

Question 9:

Differentiate the function with respect to *x*.

$$x^{\sin x} + (\sin x)^{\cos x}$$

Answer

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Also, let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \qquad ...(1)$$

$u = x^{\sin x}$

$\Rightarrow \log u = \log\left(x^{\sin x}\right)$

$\Rightarrow \log u = \sin x \log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(\sin x) \cdot \log x + \sin x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\cos x \log x + \sin x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right] \qquad \qquad ...(2)$$

$v = (\sin x)^{\cos x}$

$\Rightarrow \log v = \log(\sin x)^{\cos x}$

$\Rightarrow \log v = \cos x \log(\sin x)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(\cos x) \times \log(\sin x) + \cos x \times \frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx} = v\left[-\sin x.\log(\sin x) + \cos x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x \log \sin x + \frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x \log \sin x + \cot x \cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[\cot x \cos x - \sin x \log \sin x\right] \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{\sin x}\left(\cos x \log x + \frac{\sin x}{x}\right) + (\sin x)^{\cos x}\left[\cos x \cot x - \sin x \log \sin x\right]$$

Question 10:

Differentiate the function with respect to *x*.

$$x^{x\cos x} + \frac{x^2+1}{x^2-1}$$

Answer

Let $y = x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$

Also, let $u = x^{x\cos x}$ and $v = \dfrac{x^2+1}{x^2-1}$

$\therefore y = u+v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(1)$$

$u = x^{x\cos x}$

$$\Rightarrow \log u = \log\left(x^{x\cos x}\right)$$

$$\Rightarrow \log u = x\cos x \log x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \cdot \cos x \cdot \log x + x \cdot \frac{d}{dx}(\cos x) \cdot \log x + x \cos x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \cdot \cos x \cdot \log x + x \cdot (-\sin x)\log x + x\cos x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}(\cos x \log x - x\sin x \log x + \cos x)$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}\left[\cos x(1+\log x) - x\sin x \log x\right] \qquad \ldots(2)$$

$$v = \frac{x^2+1}{x^2-1}$$

$$\Rightarrow \log v = \log(x^2+1) - \log(x^2-1)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{2x}{x^2+1} - \frac{2x}{x^2-1}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{2x(x^2-1) - 2x(x^2+1)}{(x^2+1)(x^2-1)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{x^2+1}{x^2-1} \times \left[\frac{-4x}{(x^2+1)(x^2-1)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{(x^2-1)^2} \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{x\cos x}\left[\cos x(1+\log x) - x\sin x \log x\right] - \frac{4x}{(x^2-1)^2}$$

Question 11:

Differentiate the function with respect to *x*.

$$(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$$

Answer

Let $y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$

Also, let $u = (x \cos x)^x$ and $v = (x \sin x)^{\frac{1}{x}}$

$\therefore y = u + v$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(1)

$u = (x \cos x)^x$

$\Rightarrow \log u = \log (x \cos x)^x$

$\Rightarrow \log u = x \log (x \cos x)$

$\Rightarrow \log u = x [\log x + \log \cos x]$

$\Rightarrow \log u = x \log x + x \log \cos x$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{u} \dfrac{du}{dx} = \dfrac{d}{dx} (x \log x) + \dfrac{d}{dx} (x \log \cos x)$

$\Rightarrow \dfrac{du}{dx} = u \left[\left\{ \log x \cdot \dfrac{d}{dx} (x) + x \cdot \dfrac{d}{dx} (\log x) \right\} + \left\{ \log \cos x \cdot \dfrac{d}{dx} (x) + x \cdot \dfrac{d}{dx} (\log \cos x) \right\} \right]$

$\Rightarrow \dfrac{du}{dx} = (x \cos x)^x \left[\left(\log x \cdot 1 + x \cdot \dfrac{1}{x} \right) + \left\{ \log \cos x \cdot 1 + x \cdot \dfrac{1}{\cos x} \cdot \dfrac{d}{dx} (\cos x) \right\} \right]$

$\Rightarrow \dfrac{du}{dx} = (x \cos x)^x \left[(\log x + 1) + \left\{ \log \cos x + \dfrac{x}{\cos x} \cdot (-\sin x) \right\} \right]$

$\Rightarrow \dfrac{du}{dx} = (x \cos x)^x \left[(1 + \log x) + (\log \cos x - x \tan x) \right]$

$\Rightarrow \dfrac{du}{dx} = (x \cos x)^x \left[1 - x \tan x + (\log x + \log \cos x) \right]$

$\Rightarrow \dfrac{du}{dx} = (x \cos x)^x \left[1 - x \tan x + \log (x \cos x) \right]$ ...(2)

$$v = (x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \log (x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x} \log (x \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} (\log x + \log \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} \log x + \frac{1}{x} \log \sin x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v} \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{x} \log x \right) + \frac{d}{dx} \left[\frac{1}{x} \log (\sin x) \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \left[\log x \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} (\log x) \right] + \left[\log (\sin x) \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} \{ \log (\sin x) \} \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \left[\log x \cdot \left(-\frac{1}{x^2} \right) + \frac{1}{x} \cdot \frac{1}{x} \right] + \left[\log (\sin x) \cdot \left(-\frac{1}{x^2} \right) + \frac{1}{x} \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \frac{1}{x^2} (1 - \log x) + \left[-\frac{\log (\sin x)}{x^2} + \frac{1}{x \sin x} \cdot \cos x \right]$$

$$\Rightarrow \frac{dv}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log x}{x^2} + \frac{-\log (\sin x) + x \cot x}{x^2} \right]$$

$$\Rightarrow \frac{dv}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log x - \log (\sin x) + x \cot x}{x^2} \right]$$

$$\Rightarrow \frac{dv}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log (x \sin x) + x \cot x}{x^2} \right] \qquad \ldots (3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (x \cos x)^x \left[1 - x \tan x + \log (x \cos x) \right] + (x \sin x)^{\frac{1}{x}} \left[\frac{x \cot x + 1 - \log (x \sin x)}{x^2} \right]$$

Question 12:

Find $\dfrac{dy}{dx}$ of function.

$x^y + y^x = 1$

Answer

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad\qquad\qquad ...(1)$$

$u = x^y$

$\Rightarrow \log u = \log\left(x^y\right)$

$\Rightarrow \log u = y \log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}\left(\log x\right)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) \qquad\qquad ...(2)$$

$v = y^x$

$\Rightarrow \log v = \log\left(y^x\right)$

$\Rightarrow \log v = x \log y$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}\left(x\right) + x \cdot \frac{d}{dx}\left(\log y\right)$$

$$\Rightarrow \frac{dv}{dx} = v\left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad\qquad ...(3)$$

From (1), (2), and (3), we obtain

$$x^y \left(\log x \frac{dy}{dx} + \frac{y}{x} \right) + y^x \left(\log y + \frac{x}{y} \frac{dy}{dx} \right) = 0$$

$$\Rightarrow \left(x^y \log x + xy^{x-1} \right) \frac{dy}{dx} = -\left(yx^{y-1} + y^x \log y \right)$$

$$\therefore \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

Question 13:

Find $\frac{dy}{dx}$ of function.

$y^x = x^y$

Answer

The given function is $y^x = x^y$

Taking logarithm on both the sides, we obtain

$x \log y = y \log x$

Differentiating both sides with respect to *x*, we obtain

$$\log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y) = \log x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + y \cdot \frac{1}{x}$$

$$\Rightarrow \log y + \frac{x}{y}\frac{dy}{dx} = \log x \frac{dy}{dx} + \frac{y}{x}$$

$$\Rightarrow \left(\frac{x}{y} - \log x \right) \frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \left(\frac{x - y \log x}{y} \right) \frac{dy}{dx} = \frac{y - x \log y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} \left(\frac{y - x \log y}{x - y \log x} \right)$$

Question 14:

Find $\frac{dy}{dx}$ of function.

$$(\cos x)^y = (\cos y)^x$$

Answer

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$$y \log \cos x = x \log \cos y$$

Differentiating both sides, we obtain

$$\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) = \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) = \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) = \log \cos y + \frac{x}{\cos y}(-\sin y) \cdot \frac{dy}{dx}$$

$$\Rightarrow \log \cos x \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$$

$$\Rightarrow (\log \cos x + x \tan y)\frac{dy}{dx} = y \tan x + \log \cos y$$

$$\therefore \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$$

Question 15:

Find $\frac{dy}{dx}$ of function.

$$xy = e^{(x-y)}$$

Answer

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\log(xy) = \log(e^{x-y})$$

$$\Rightarrow \log x + \log y = (x - y)\log e$$

$$\Rightarrow \log x + \log y = (x - y) \times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \left(1 + \frac{1}{y}\right)\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence

find $f'(1)$.

Answer

The given relationship is $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{f(x)} \cdot \frac{d}{dx}[f(x)] = \frac{d}{dx}\log(1+x) + \frac{d}{dx}\log(1+x^2) + \frac{d}{dx}\log(1+x^4) + \frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow \frac{1}{f(x)} \cdot f'(x) = \frac{1}{1+x}\cdot\frac{d}{dx}(1+x) + \frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2) + \frac{1}{1+x^4}\cdot\frac{d}{dx}(1+x^4) + \frac{1}{1+x^8}\cdot\frac{d}{dx}(1+x^8)$$

$$\Rightarrow f'(x) = f(x)\left[\frac{1}{1+x} + \frac{1}{1+x^2}\cdot 2x + \frac{1}{1+x^4}\cdot 4x^3 + \frac{1}{1+x^8}\cdot 8x^7\right]$$

$$\therefore f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Hence, $f'(1) = (1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1} + \frac{2\times1}{1+1^2} + \frac{4\times1^3}{1+1^4} + \frac{8\times1^7}{1+1^8}\right]$

$$= 2\times2\times2\times2\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right]$$

$$= 16\times\left(\frac{1+2+4+8}{2}\right)$$

$$= 16\times\frac{15}{2} = 120$$

Question 17:

Differentiate $(x^5 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below

(i) By using product rule.

(ii) By expanding the product to obtain a single polynomial.

(iii By logarithmic differentiation.

Do they all give the same answer?

Answer

(i) Let $y = (x^5 - 5x + 8)(x^3 + 7x + 9)$

Let $x^2 - 5x + 8 = u$ and $x^3 + 7x + 9 = v$

$\therefore y = uv$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} \cdot v + u \cdot \dfrac{dv}{dx}$ (By using product rule)

$\Rightarrow \dfrac{dy}{dx} = \dfrac{d}{dx}(x^2 - 5x + 8) \cdot (x^3 + 7x + 9) + (x^2 - 5x + 8) \cdot \dfrac{d}{dx}(x^3 + 7x + 9)$

$\Rightarrow \dfrac{dy}{dx} = (2x - 5)(x^3 + 7x + 9) + (x^2 - 5x + 8)(3x^2 + 7)$

$\Rightarrow \dfrac{dy}{dx} = 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + x^2(3x^2 + 7) - 5x(3x^2 + 7) + 8(3x^2 + 7)$

$\Rightarrow \dfrac{dy}{dx} = (2x^4 + 14x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 + 7x^2) - 15x^3 - 35x + 24x^2 + 56$

$\therefore \dfrac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$

(ii) $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

$\qquad = x^2(x^3 + 7x + 9) - 5x(x^3 + 7x + 9) + 8(x^3 + 7x + 9)$

$\qquad = x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$

$\qquad = x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$

$\therefore \dfrac{dy}{dx} = \dfrac{d}{dx}(x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72)$

$\qquad = \dfrac{d}{dx}(x^5) - 5\dfrac{d}{dx}(x^4) + 15\dfrac{d}{dx}(x^3) - 26\dfrac{d}{dx}(x^2) + 11\dfrac{d}{dx}(x) + \dfrac{d}{dx}(72)$

$\qquad = 5x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0$

$\qquad = 5x^4 - 20x^3 + 45x^2 - 52x + 11$

(iii) $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

Taking logarithm on both the sides, we obtain

$\log y = \log(x^2 - 5x + 8) + \log(x^3 + 7x + 9)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\log\left(x^2 - 5x + 8\right) + \frac{d}{dx}\log\left(x^3 + 7x + 9\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8} \cdot \frac{d}{dx}\left(x^2 - 5x + 8\right) + \frac{1}{x^3 + 7x + 9} \cdot \frac{d}{dx}\left(x^3 + 7x + 9\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x^2 - 5x + 8} \times (2x - 5) + \frac{1}{x^3 + 7x + 9} \times \left(3x^2 + 7\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)\left[\frac{2x - 5}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)\left[\frac{(2x - 5)\left(x^3 + 7x + 9\right) + \left(3x^2 + 7\right)\left(x^2 - 5x + 8\right)}{\left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)}\right]$$

$$\Rightarrow \frac{dy}{dx} = 2x\left(x^3 + 7x + 9\right) - 5\left(x^3 + 7x + 9\right) + 3x^2\left(x^2 - 5x + 8\right) + 7\left(x^2 - 5x + 8\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(2x^4 + 14x^2 + 18x\right) - 5x^3 - 35x - 45 + \left(3x^4 - 15x^3 + 24x^2\right) + \left(7x^2 - 35x + 56\right)$$

$$\Rightarrow \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

Question 18:

If *u*, *v* and *w* are functions of *x*, then show that

$$\frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Answer

Let $y = u.v.w = u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx} = \frac{du}{dx}.(v.w) + u.\frac{d}{dx}(v \cdot w)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} v \cdot w + u\left[\frac{dv}{dx} \cdot w + v \cdot \frac{dw}{dx}\right] \qquad (\text{Again applying product rule})$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y = u.v.w$, we obtain

$$\log y = \log u + \log v + \log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w.\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

Exercise 5.6

Question 1:

If x and y are connected parametrically by the equation, without eliminating the

parameter, find $\dfrac{dy}{dx}$.

$x = 2at^2$, $y = at^4$

Answer

The given equations are $x = 2at^2$ and $y = at^4$

Then, $\dfrac{dx}{dt} = \dfrac{d}{dt}\left(2at^2\right) = 2a \cdot \dfrac{d}{dt}\left(t^2\right) = 2a \cdot 2t = 4at$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\left(at^4\right) = a \cdot \dfrac{d}{dt}\left(t^4\right) = a \cdot 4 \cdot t^3 = 4at^3$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \dfrac{4at^3}{4at} = t^2$

Question 2:

If x and y are connected parametrically by the equation, without eliminating the

parameter, find $\dfrac{dy}{dx}$.

$x = a \cos \theta$, $y = b \cos \theta$

Answer

The given equations are $x = a \cos \theta$ and $y = b \cos \theta$

Then, $\dfrac{dx}{d\theta} = \dfrac{d}{d\theta}(a\cos\theta) = a(-\sin\theta) = -a\sin\theta$

$\dfrac{dy}{d\theta} = \dfrac{d}{d\theta}(b\cos\theta) = b(-\sin\theta) = -b\sin\theta$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \dfrac{-b\sin\theta}{-a\sin\theta} = \dfrac{b}{a}$

Question 3:

If *x* and *y* are connected parametrically by the equation, without eliminating the

parameter, find $\dfrac{dy}{dx}$.

x = sin *t*, *y* = cos 2*t*

Answer

The given equations are *x* = sin *t* and *y* = cos 2*t*

Then, $\dfrac{dx}{dt} = \dfrac{d}{dt}\left(\sin t\right) = \cos t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\left(\cos 2t\right) = -\sin 2t \cdot \dfrac{d}{dt}\left(2t\right) = -2\sin 2t$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \dfrac{-2\sin 2t}{\cos t} = \dfrac{-2 \cdot 2\sin t \cos t}{\cos t} = -4\sin t$

Question 4:

If *x* and *y* are connected parametrically by the equation, without eliminating the

parameter, find $\dfrac{dy}{dx}$.

$x = 4t,\ y = \dfrac{4}{t}$

Answer

The given equations are $x = 4t$ and $y = \dfrac{4}{t}$

$$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4 \cdot \frac{d}{dt}\left(\frac{1}{t}\right) = 4 \cdot \left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{\left(\dfrac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

Question 5:

If *x* and *y* are connected parametrically by the equation, without eliminating the

parameter, find $\dfrac{dy}{dx}$

$x = \cos\theta - \cos 2\theta, \; y = \sin\theta - \sin 2\theta$

Answer

The given equations are $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

Then, $\dfrac{dx}{d\theta} = \dfrac{d}{d\theta}(\cos\theta - \cos 2\theta) = \dfrac{d}{d\theta}(\cos\theta) - \dfrac{d}{d\theta}(\cos 2\theta)$

$$= -\sin\theta - (-2\sin 2\theta) = 2\sin 2\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \sin 2\theta) = \frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\sin 2\theta)$$

$$= \cos\theta - 2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

Question 6:

If *x* and *y* are connected parametrically by the equation, without eliminating the

parameter, find $\dfrac{dy}{dx}$

$x = a(\theta - \sin\theta), \; y = a(1 + \cos\theta)$

Answer

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

$$\text{Then,} \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a\left[0 + (-\sin\theta)\right] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1 - \cos\theta)} = \frac{-2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{2\sin^2\dfrac{\theta}{2}} = \frac{-\cos\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Question 7:

If *x* and *y* are connected parametrically by the equation, without eliminating the

parameter, find $\dfrac{dy}{dx}$

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, \quad y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Answer

The given equations are $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$

Then, $\dfrac{dx}{dt} = \dfrac{d}{dt}\left[\dfrac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$= \dfrac{\sqrt{\cos 2t} \cdot \dfrac{d}{dt}\left(\sin^3 t\right) - \sin^3 t \cdot \dfrac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \dfrac{\sqrt{\cos 2t} \cdot 3\sin^2 t \cdot \dfrac{d}{dt}\left(\sin t\right) - \sin^3 t \times \dfrac{1}{2\sqrt{\cos 2t}} \cdot \dfrac{d}{dt}\left(\cos 2t\right)}{\cos 2t}$$

$$= \dfrac{3\sqrt{\cos 2t} \cdot \sin^2 t \cos t - \dfrac{\sin^3 t}{2\sqrt{\cos 2t}} \cdot \left(-2\sin 2t\right)}{\cos 2t}$$

$$= \dfrac{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\left[\dfrac{\cos^3 t}{\sqrt{\cos 2t}}\right]$

$$= \dfrac{\sqrt{\cos 2t}. \dfrac{d}{dt}\left(\cos^3 t\right) - \cos^3 t \cdot \dfrac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$= \dfrac{\sqrt{\cos 2t}.3\cos^2 t \cdot \dfrac{d}{dt}\left(\cos t\right) - \cos^3 t. \dfrac{1}{2\sqrt{\cos 2t}} \cdot \dfrac{d}{dt}\left(\cos 2t\right)}{\cos 2t}$$

$$= \dfrac{3\sqrt{\cos 2t}.\cos^2 t\left(-\sin t\right) - \cos^3 t \cdot \dfrac{1}{2\sqrt{\cos 2t}}. \left(-2\sin 2t\right)}{\cos 2t}$$

$$= \dfrac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \sin 2t}{\cos 2t \cdot \sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \sin 2t}{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \left(2\sin t \cos t\right)}{3\cos 2t \sin^2 t \cos t + \sin^3 t \left(2\sin t \cos t\right)}$$

$$= \frac{\sin t \cos t \left[-3\cos 2t.\cos t + 2\cos^3 t\right]}{\sin t \cos t \left[3\cos 2t \sin t + 2\sin^3 t\right]}$$

$$= \frac{\left[-3\left(2\cos^2 t - 1\right)\cos t + 2\cos^3 t\right]}{\left[3\left(1 - 2\sin^2 t\right)\sin t + 2\sin^3 t\right]} \qquad \begin{bmatrix} \cos 2t = \left(2\cos^2 t - 1\right), \\ \cos 2t = \left(1 - 2\sin^2 t\right) \end{bmatrix}$$

$$= \frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t}$$

$$= \frac{-\cos 3t}{\sin 3t} \qquad \begin{bmatrix} \cos 3t = 4\cos^3 t - 3\cos t, \\ \sin 3t = 3\sin t - 4\sin^3 t \end{bmatrix}$$

$$= -\cot 3t$$

Question 8:

If *x* and *y* are connected parametrically by the equation, without eliminating the

parameter, find $\dfrac{dy}{dx}$.

$$x = a\left(\cos t + \log \tan \frac{t}{2}\right), \ y = a\sin t$$

Answer

The given equations are $x = a\left(\cos t + \log \tan \dfrac{t}{2}\right)$ and $y = a\sin t$

Then, $\dfrac{dx}{dt} = a \cdot \left[\dfrac{d}{dt}(\cos t) + \dfrac{d}{dt}\left(\log \tan \dfrac{t}{2}\right) \right]$

$= a\left[-\sin t + \dfrac{1}{\tan \dfrac{t}{2}} \cdot \dfrac{d}{dt}\left(\tan \dfrac{t}{2}\right) \right]$

$= a\left[-\sin t + \cot \dfrac{t}{2} \cdot \sec^2 \dfrac{t}{2} \cdot \dfrac{d}{dt}\left(\dfrac{t}{2}\right) \right]$

$= a\left[-\sin t + \dfrac{\cos \dfrac{t}{2}}{\sin \dfrac{t}{2}} \times \dfrac{1}{\cos^2 \dfrac{t}{2}} \times \dfrac{1}{2} \right]$

$= a\left[-\sin t + \dfrac{1}{2\sin \dfrac{t}{2}\cos \dfrac{t}{2}} \right]$

$= a\left(-\sin t + \dfrac{1}{\sin t} \right)$

$= a\left(\dfrac{-\sin^2 t + 1}{\sin t} \right)$

$= a\dfrac{\cos^2 t}{\sin t}$

$\dfrac{dy}{dt} = a\dfrac{d}{dt}(\sin t) = a\cos t$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \dfrac{a\cos t}{a\dfrac{\cos^2 t}{\sin t}} = \dfrac{\sin t}{\cos t} = \tan t$

Question 9:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\dfrac{dy}{dx}$.

$x = a\sec\theta,\ y = b\tan\theta$

Answer

The given equations are $x = a\sec\theta$ and $y = b\tan\theta$

Then, $\dfrac{dx}{d\theta} = a \cdot \dfrac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$

$\dfrac{dy}{d\theta} = b \cdot \dfrac{d}{d\theta}(\tan\theta) = b\sec^2\theta$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \dfrac{b\sec^2\theta}{a\sec\theta\tan\theta} = \dfrac{b}{a}\sec\theta\cot\theta = \dfrac{b\cos\theta}{a\cos\theta\sin\theta} = \dfrac{b}{a} \times \dfrac{1}{\sin\theta} = \dfrac{b}{a}\operatorname{cosec}\theta$

Question 10:

If *x* and *y* are connected parametrically by the equation, without eliminating the

parameter, find $\dfrac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

Answer

The given equations are $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

Then, $\dfrac{dx}{d\theta} = a\left[\dfrac{d}{d\theta}\cos\theta + \dfrac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\dfrac{d}{d\theta}(\sin\theta) + \sin\theta\dfrac{d}{d\theta}(\theta)\right]$

$= a\left[-\sin\theta + \theta\cos\theta + \sin\theta\right] = a\theta\cos\theta$

$\dfrac{dy}{d\theta} = a\left[\dfrac{d}{d\theta}(\sin\theta) - \dfrac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\dfrac{d}{d\theta}(\cos\theta) + \cos\theta \cdot \dfrac{d}{d\theta}(\theta)\right\}\right]$

$= a\left[\cos\theta + \theta\sin\theta - \cos\theta\right]$

$= a\theta\sin\theta$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \dfrac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$

Question 11:

If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, show that $\dfrac{dy}{dx} = -\dfrac{y}{x}$

Answer

The given equations are $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

$x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

$\Rightarrow x = \left(a^{\sin^{-1}t}\right)^{\frac{1}{2}}$ and $y = \left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$

$\Rightarrow x = a^{\frac{1}{2}\sin^{-1}t}$ and $y = a^{\frac{1}{2}\cos^{-1}t}$

Consider $x = a^{\frac{1}{2}\sin^{-1}t}$

Taking logarithm on both the sides, we obtain

$\log x = \dfrac{1}{2}\sin^{-1}t \log a$

$\therefore \dfrac{1}{x}\cdot\dfrac{dx}{dt} = \dfrac{1}{2}\log a \cdot \dfrac{d}{dt}\left(\sin^{-1}t\right)$

$\Rightarrow \dfrac{dx}{dt} = \dfrac{x}{2}\log a \cdot \dfrac{1}{\sqrt{1-t^2}}$

$\Rightarrow \dfrac{dx}{dt} = \dfrac{x\log a}{2\sqrt{1-t^2}}$

Then, consider $y = a^{\frac{1}{2}\cos^{-1}t}$

Taking logarithm on both the sides, we obtain

$\log y = \dfrac{1}{2}\cos^{-1}t \log a$

$\therefore \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{1}{2}\log a \cdot \dfrac{d}{dt}\left(\cos^{-1}t\right)$

$\Rightarrow \dfrac{dy}{dt} = \dfrac{y\log a}{2}\cdot\left(\dfrac{-1}{\sqrt{1-t^2}}\right)$

$\Rightarrow \dfrac{dy}{dt} = \dfrac{-y\log a}{2\sqrt{1-t^2}}$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{\left(\dfrac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\dfrac{x\log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}.$$

Hence, proved.

Exercise 5.7

Question 1:

Find the second order derivatives of the function.

$x^2 + 3x + 2$

Answer

Let $y = x^2 + 3x + 2$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x+3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

Question 2:

Find the second order derivatives of the function.

x^{20}

Answer

Let $y = x^{20}$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x^{20}) = 20x^{19}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19}) = 20\frac{d}{dx}(x^{19}) = 20 \cdot 19 \cdot x^{18} = 380x^{18}$$

Question 3:

Find the second order derivatives of the function.

$x \cdot \cos x$

Answer

Let $y = x \cdot \cos x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot \cos x) = \cos x \cdot \frac{d}{dx}(x) + x\frac{d}{dx}(\cos x) = \cos x \cdot 1 + x(-\sin x) = \cos x - x\sin x$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}[\cos x - x\sin x] = \frac{d}{dx}(\cos x) - \frac{d}{dx}(x\sin x)$$

$$= -\sin x - \left[\sin x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin x)\right]$$

$$= -\sin x - (\sin x + x\cos x)$$

$$= -(x\cos x + 2\sin x)$$

Question 4:

Find the second order derivatives of the function.

$\log x$

Answer

Let $y = \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

Question 5:

Find the second order derivatives of the function.

$x^3 \log x$

Answer

Let $y = x^3 \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^3 \log x\right] = \log x \cdot \frac{d}{dx}\left(x^3\right) + x^3 \cdot \frac{d}{dx}\left(\log x\right)$$

$$= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2$$

$$= x^2 \left(1 + 3\log x\right)$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left[x^2 \left(1 + 3\log x\right)\right]$$

$$= \left(1 + 3\log x\right) \cdot \frac{d}{dx}\left(x^2\right) + x^2 \frac{d}{dx}\left(1 + 3\log x\right)$$

$$= \left(1 + 3\log x\right) \cdot 2x + x^2 \cdot \frac{3}{x}$$

$$= 2x + 6x \log x + 3x$$

$$= 5x + 6x \log x$$

$$= x\left(5 + 6\log x\right)$$

Question 6:

Find the second order derivatives of the function.

$e^x \sin 5x$

Answer

Let $y = e^x \sin 5x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x \sin 5x\right) = \sin 5x \cdot \frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}\left(\sin 5x\right)$$

$$= \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx}\left(5x\right) = e^x \sin 5x + e^x \cos 5x \cdot 5$$

$$= e^x \left(\sin 5x + 5\cos 5x\right)$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left[e^x \left(\sin 5x + 5\cos 5x\right)\right]$$

$$= \left(\sin 5x + 5\cos 5x\right) \cdot \frac{d}{dx}\left(e^x\right) + e^x \cdot \frac{d}{dx}\left(\sin 5x + 5\cos 5x\right)$$

$$= \left(\sin 5x + 5\cos 5x\right)e^x + e^x \left[\cos 5x \cdot \frac{d}{dx}\left(5x\right) + 5\left(-\sin 5x\right) \cdot \frac{d}{dx}\left(5x\right)\right]$$

$$= e^x \left(\sin 5x + 5\cos 5x\right) + e^x \left(5\cos 5x - 25\sin 5x\right)$$

$$= e^x \left(10\cos 5x - 24\sin 5x\right) = 2e^x \left(5\cos 5x - 12\sin 5x\right)$$

Then,

Question 7:

Find the second order derivatives of the function.

$e^{6x} \cos 3x$

Answer

Let $y = e^{6x} \cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x} \cdot \cos 3x\right) = \cos 3x \cdot \frac{d}{dx}\left(e^{6x}\right) + e^{6x} \cdot \frac{d}{dx}\left(\cos 3x\right)$$

$$= \cos 3x \cdot e^{6x} \cdot \frac{d}{dx}\left(6x\right) + e^{6x} \cdot \left(-\sin 3x\right) \cdot \frac{d}{dx}\left(3x\right)$$

$$= 6e^{6x} \cos 3x - 3e^{6x} \sin 3x \qquad \ldots(1)$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(6e^{6x} \cos 3x - 3e^{6x} \sin 3x\right) = 6 \cdot \frac{d}{dx}\left(e^{6x} \cos 3x\right) - 3 \cdot \frac{d}{dx}\left(e^{6x} \sin 3x\right)$$

$$= 6 \cdot \left[6e^{6x} \cos 3x - 3e^{6x} \sin 3x\right] - 3 \cdot \left[\sin 3x \cdot \frac{d}{dx}\left(e^{6x}\right) + e^{6x} \cdot \frac{d}{dx}\left(\sin 3x\right)\right] \qquad \left[\text{Using } (1)\right]$$

$$= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 3\left[\sin 3x \cdot e^{6x} \cdot 6 + e^{6x} \cdot \cos 3x \cdot 3\right]$$

$$= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 18e^{6x} \sin 3x - 9e^{6x} \cos 3x$$

$$= 27e^{6x} \cos 3x - 36e^{6x} \sin 3x$$

$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

Question 8:

Find the second order derivatives of the function.

$\tan^{-1} x$

Answer

Let $y = \tan^{-1} x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{d}{dx}\left(1+x^2\right)^{-1} = (-1)\cdot\left(1+x^2\right)^{-2}\cdot\frac{d}{dx}\left(1+x^2\right)$$

$$= \frac{-1}{\left(1+x^2\right)^2} \times 2x = \frac{-2x}{\left(1+x^2\right)^2}$$

Question 9:

Find the second order derivatives of the function.

$$\log\left(\log x\right)$$

Answer

Let $y = \log\left(\log x\right)$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\log x\right)\right] = \frac{1}{\log x}\cdot\frac{d}{dx}\left(\log x\right) = \frac{1}{x\log x} = \left(x\log x\right)^{-1}$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left[\left(x\log x\right)^{-1}\right] = (-1)\cdot\left(x\log x\right)^{-2}\cdot\frac{d}{dx}\left(x\log x\right)$$

$$= \frac{-1}{\left(x\log x\right)^2}\cdot\left[\log x\cdot\frac{d}{dx}\left(x\right) + x\cdot\frac{d}{dx}\left(\log x\right)\right]$$

$$= \frac{-1}{\left(x\log x\right)^2}\cdot\left[\log x\cdot 1 + x\cdot\frac{1}{x}\right] = \frac{-\left(1+\log x\right)}{\left(x\log x\right)^2}$$

Question 10:

Find the second order derivatives of the function.

$$\sin\left(\log x\right)$$

Answer

Let $y = \sin\left(\log x\right)$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\Big[\sin(\log x)\Big] = \cos(\log x) \cdot \frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$= \frac{x \cdot \frac{d}{dx}\Big[\cos(\log x)\Big] - \cos(\log x) \cdot \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x \cdot \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x)\right] - \cos(\log x).1}{x^2}$$

$$= \frac{-x\sin(\log x).\frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-\Big[\sin(\log x) + \cos(\log x)\Big]}{x^2}$$

Question 11:

If $y = 5\cos x - 3\sin x$, prove that $\dfrac{d^2y}{dx^2} + y = 0$

Answer

It is given that, $y = 5\cos x - 3\sin x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x) = 5\frac{d}{dx}(\cos x) - 3\frac{d}{dx}(\sin x)$$

$$= 5(-\sin x) - 3\cos x = -(5\sin x + 3\cos x)$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left[-(5\sin x + 3\cos x)\right]$$

$$= -\left[5\cdot\frac{d}{dx}(\sin x) + 3\cdot\frac{d}{dx}(\cos x)\right]$$

$$= -\left[5\cos x + 3(-\sin x)\right]$$

$$= -\left[5\cos x - 3\sin x\right]$$

$$= -y$$

$$\therefore \frac{d^2 y}{dx^2} + y = 0$$

Hence, proved.

Question 12:

If $y = \cos^{-1} x$, find $\dfrac{d^2 y}{dx^2}$ in terms of y alone.

Answer

It is given that, $y = \cos^{-1} x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}} = -\left(1-x^2\right)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$= -\left(-\frac{1}{2}\right)\cdot\left(1-x^2\right)^{\frac{-3}{2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$= \frac{1}{2\sqrt{\left(1-x^2\right)^3}}\times\left(-2x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x}{\sqrt{\left(1-x^2\right)^3}} \qquad \ldots(i)$$

$$y = \cos^{-1}x \Rightarrow x = \cos y$$

Putting $x = \cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$= \frac{-\cos y}{\sin^3 y}$$

$$= \frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\cot y\cdot\operatorname{cosec}^2 y$$

Question 13:

If $y = 3\cos\left(\log x\right) + 4\sin\left(\log x\right)$, show that $x^2 y_2 + xy_1 + y = 0$

Answer

It is given that, $y = 3\cos\left(\log x\right) + 4\sin\left(\log x\right)$

Then,

$$y_1 = 3 \cdot \frac{d}{dx}\left[\cos(\log x)\right] + 4 \cdot \frac{d}{dx}\left[\sin(\log x)\right]$$

$$= 3 \cdot \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x)\right] + 4 \cdot \left[\cos(\log x) \cdot \frac{d}{dx}(\log x)\right]$$

$$\therefore y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

$$\therefore y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x\{4\cos(\log x) - 3\sin(\log x)\}' - \{4\cos(\log x) - 3\sin(\log x)\}(x)'}{x^2}$$

$$= \frac{x\left[4\{\cos(\log x)\}' - 3\{\sin(\log x)\}'\right] - \{4\cos(\log x) - 3\sin(\log x)\}.1}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).\frac{1}{x} - 3\cos(\log x).\frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$$\therefore x^2 y_2 + x y_1 + y$$

$$= x^2 \left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) + 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)$$

$$= 0$$

Hence, proved.

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\dfrac{d^2 y}{dx^2} - (m+n)\dfrac{dy}{dx} + mny = 0$

Answer

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A \cdot \frac{d}{dx}\left(e^{mx}\right) + B \cdot \frac{d}{dx}\left(e^{nx}\right) = A \cdot e^{mx} \cdot \frac{d}{dx}\left(mx\right) + B \cdot e^{nx} \cdot \frac{d}{dx}\left(nx\right) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am \cdot \frac{d}{dx}\left(e^{mx}\right) + Bn \cdot \frac{d}{dx}\left(e^{nx}\right)$$

$$= Am \cdot e^{mx} \cdot \frac{d}{dx}\left(mx\right) + Bn \cdot e^{nx} \cdot \frac{d}{dx}\left(nx\right) = Am^2 e^{mx} + Bn^2 e^{nx}$$

$$\therefore \frac{d^2 y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2 e^{mx} + Bn^2 e^{nx} - (m+n) \cdot \left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2 e^{mx} + Bn^2 e^{nx} - Am^2 e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2 e^{nx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence, proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2 y}{dx^2} = 49y$

Answer

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx} = 500.\frac{d}{dx}\left(e^{7x}\right) + 600.\frac{d}{dx}\left(e^{-7x}\right)$$

$$= 500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$

$$= 3500e^{7x} - 4200e^{-7x}$$

$$\therefore \frac{d^2y}{dx^2} = 3500 \cdot \frac{d}{dx}\left(e^{7x}\right) - 4200.\frac{d}{dx}\left(e^{-7x}\right)$$

$$= 3500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$

$$= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x}$$

$$= 49 \times 500e^{7x} + 49 \times 600e^{-7x}$$

$$= 49\left(500e^{7x} + 600e^{-7x}\right)$$

$$= 49y$$

Hence, proved.

Question 16:

If $e^y(x+1) = 1$, show that $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

Answer

The given relationship is $e^y(x+1) = 1$

$$e^y(x+1) = 1$$

$$\Rightarrow e^y = \frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y = \log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{dy}{dx} = (x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right) = (x+1).\frac{-1}{(x+1)^2} = \frac{-1}{x+1}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{1}{x+1}\right) = -\left(\frac{-1}{(x+1)^2}\right) = \frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

Hence, proved.

Question 17:

If $y = \left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2 y_2 + 2x\left(x^2+1\right)y_1 = 2$

Answer

The given relationship is $y = \left(\tan^{-1}x\right)^2$

Then,

$$y_1 = 2\tan^{-1}x\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\Rightarrow y_1 = 2\tan^{-1}x.\frac{1}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)y_1 = 2\tan^{-1}x$$

Again differentiating with respect to x on both the sides, we obtain

$$\left(1+x^2\right)y_2 + 2xy_1 = 2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right)^2 y_2 + 2x\left(1+x^2\right)y_1 = 2$$

Hence, proved.

Exercise 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$

Answer

The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in [−4, 2] and is differentiable in (−4, 2).

$$f(-4) = (-4)^2 + 2 \times (-4) - 8 = 16 - 8 - 8 = 0$$
$$f(2) = (2)^2 + 2 \times 2 - 8 = 4 + 4 - 8 = 0$$

∴ $f(-4) = f(2) = 0$

⇒ The value of $f(x)$ at −4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in (-4, 2)$ such that $f'(c) = 0$

$$f(x) = x^2 + 2x - 8$$
$$\Rightarrow f'(x) = 2x + 2$$
$$\therefore f'(c) = 0$$
$$\Rightarrow 2c + 2 = 0$$
$$\Rightarrow c = -1, \text{ where } c = -1 \in (-4, 2)$$

Hence, Rolle's Theorem is verified for the given function.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x \in [5,\ 9]$

(ii) $f(x)=[x]$ for $x \in [-2,\ 2]$

(iii) $f(x)=x^2-1$ for $x \in [1,\ 2]$

Answer

By Rolle's Theorem, for a function $f:[a,\ b] \rightarrow \mathbf{R}$, if

(a) *f* is continuous on [*a*, *b*]

(b) *f* is differentiable on (*a*, *b*)

(c) *f* (*a*) = *f* (*b*)

then, there exists some *c* ∈ (*a*, *b*) such that $f'(c)=0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x \in [5,\ 9]$

It is evident that the given function *f* (*x*) is not continuous at every integral point.
In particular, *f(x)* is not continuous at *x* = 5 and *x* = 9

⇒ *f* (*x*) is not continuous in [5, 9].

Also, $f(5)=[5]=5$ and $f(9)=[9]=9$

$\therefore f(5) \neq f(9)$

The differentiability of *f* in (5, 9) is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

∴f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point. In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

⇒ $f(x)$ is not continuous in [−2, 2].

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

∴ $f(-2) \neq f(2)$

The differentiability of f in (−2, 2) is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

$$f(1) = (1)^2 - 1 = 0$$
$$f(2) = (2)^2 - 1 = 3$$

$\therefore f(1) \neq f(2)$

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1,\ 2]$.

Question 3:

If $f : [-5,5] \rightarrow \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then

prove that $f(-5) \neq f(5)$.

Answer

It is given that $f : [-5,5] \rightarrow \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on [−5, 5].

(b) f is differentiable on (−5, 5).

Therefore, by the Mean Value Theorem, there exists $c \in$ (−5, 5) such that

$f'(c) = \dfrac{f(5) - f(-5)}{5 - (-5)}$

$\Rightarrow 10 f'(c) = f(5) - f(-5)$

It is also given that $f'(x)$ does not vanish anywhere.

$\therefore f'(c) \neq 0$

$\Rightarrow 10 f'(c) \neq 0$

$\Rightarrow f(5) - f(-5) \neq 0$

$\Rightarrow f(5) \neq f(-5)$

Hence, proved.

Question 4:

Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a,\ b]$, where

$a = 1$ and $b = 4$.

Answer

The given function is $f(x) = x^2 - 4x - 3$

f, being a polynomial function, is continuous in [1, 4] and is differentiable in (1, 4)

whose derivative is $2x - 4$.

$f(1) = 1^2 - 4 \times 1 - 3 = -6, \ f(4) = 4^2 - 4 \times 4 - 3 = -3$

$\therefore \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{-3 - (-6)}{3} = \dfrac{3}{3} = 1$

Mean Value Theorem states that there is a point $c \in$ (1, 4) such that $f'(c) = 1$

$f'(c) = 1$

$\Rightarrow 2c - 4 = 1$

$\Rightarrow c = \dfrac{5}{2}$, where $c = \dfrac{5}{2} \in (1, \ 4)$

Hence, Mean Value Theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval [a, b], where a = 1 and

b = 3. Find all $c \in (1,3)$ for which $f'(c) = 0$

Answer

The given function f is $f(x) = x^3 - 5x^2 - 3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3)

whose derivative is $3x^2 - 10x - 3$.

$f(1) = 1^3 - 5 \times 1^2 - 3 \times 1 = -7, \ f(3) = 3^3 - 5 \times 3^2 - 3 \times 3 = -27$

$\therefore \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{-27 - (-7)}{3 - 1} = -10$

Mean Value Theorem states that there exist a point $c \in (1, 3)$ such that $f'(c) = -10$

$f'(c) = -10$

$\Rightarrow 3c^2 - 10c - 3 = 10$

$\Rightarrow 3c^2 - 10c + 7 = 0$

$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$

$\Rightarrow 3c(c-1) - 7(c-1) = 0$

$\Rightarrow (c-1)(3c-7) = 0$

$\Rightarrow c = 1, \dfrac{7}{3}$, where $c = \dfrac{7}{3} \in (1, 3)$

Hence, Mean Value Theorem is verified for the given function and $c = \dfrac{7}{3} \in (1, 3)$ is the

only point for which $f'(c) = 0$

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given in the

above exercise 2.

Answer

Mean Value Theorem states that for a function $f : [a, b] \rightarrow \mathbf{R}$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

then, there exists some $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy

any of the two conditions of the hypothesis.

(i) $f(x)=[x]$ for $x \in [5, 9]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, $f(x)$ is not continuous at x = 5 and x = 9

⇒ f (x) is not continuous in [5, 9].

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that $n \in$ (5, 9).

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x \in [5, 9]$.

(ii) $f(x)=[x]$ for $x \in [-2, 2]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x \in [-2, 2]$.

(iii) $f(x)=x^2 -1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

It can be proved as follows.

$f(1) = 1^2 - 1 = 0,\ f(2) = 2^2 - 1 = 3$

$\therefore \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{3 - 0}{1} = 3$

$f'(x) = 2x$

$\therefore f'(c) = 3$

$\Rightarrow 2c = 3$

$\Rightarrow c = \dfrac{3}{2} = 1.5,\ \text{where } 1.5 \in [1, 2]$

Miscellaneous Solutions

Question 1:

$\left(3x^2 - 9x + 5\right)^9$

Answer

Let $y = \left(3x^2 - 9x + 5\right)^9$

Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^2 - 9x + 5\right)^9$$

$$= 9\left(3x^2 - 9x + 5\right)^8 \cdot \frac{d}{dx}\left(3x^2 - 9x + 5\right)$$

$$= 9\left(3x^2 - 9x + 5\right)^8 \cdot \left(6x - 9\right)$$

$$= 9\left(3x^2 - 9x + 5\right)^8 \cdot 3\left(2x - 3\right)$$

$$= 27\left(3x^2 - 9x + 5\right)^8 \left(2x - 3\right)$$

Question 2:

$\sin^3 x + \cos^6 x$

Answer

Let $y = \sin^3 x + \cos^6 x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\sin^3 x\right) + \frac{d}{dx}\left(\cos^6 x\right)$$

$$= 3\sin^2 x \cdot \frac{d}{dx}\left(\sin x\right) + 6\cos^5 x \cdot \frac{d}{dx}\left(\cos x\right)$$

$$= 3\sin^2 x \cdot \cos x + 6\cos^5 x \cdot \left(-\sin x\right)$$

$$= 3\sin x \cos x \left(\sin x - 2\cos^4 x\right)$$

Question 3:

$\left(5x\right)^{3\cos 2x}$

Answer

Let $y = (5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y = 3\cos 2x \log 5x$

Differentiating both sides with respect to *x*, we obtain

$\dfrac{1}{y}\dfrac{dy}{dx} = 3\left[\log 5x \cdot \dfrac{d}{dx}(\cos 2x) + \cos 2x \cdot \dfrac{d}{dx}(\log 5x)\right]$

$\Rightarrow \dfrac{dy}{dx} = 3y\left[\log 5x(-\sin 2x) \cdot \dfrac{d}{dx}(2x) + \cos 2x \cdot \dfrac{1}{5x} \cdot \dfrac{d}{dx}(5x)\right]$

$\Rightarrow \dfrac{dy}{dx} = 3y\left[-2\sin 2x \log 5x + \dfrac{\cos 2x}{x}\right]$

$\Rightarrow \dfrac{dy}{dx} = 3y\left[\dfrac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$

$\therefore \dfrac{dy}{dx} = (5x)^{3\cos 2x}\left[\dfrac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$

Question 4:

$\sin^{-1}\left(x\sqrt{x}\right),\ 0 \le x \le 1$

Answer

Let $y = \sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}} \times \frac{d}{dx}\left(x\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2} \cdot x^{\frac{1}{2}}$$

$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Question 5:

$$\frac{\cos^{-1}\dfrac{x}{2}}{\sqrt{2x+7}}, \quad -2 < x < 2$$

Answer

Let $y = \dfrac{\cos^{-1}\dfrac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\,\dfrac{d}{dx}\left(\cos^{-1}\dfrac{x}{2}\right) - \left(\cos^{-1}\dfrac{x}{2}\right)\dfrac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$= \frac{\sqrt{2x+7}\left[\dfrac{-1}{\sqrt{1-\left(\dfrac{x}{2}\right)^2}}\cdot\dfrac{d}{dx}\left(\dfrac{x}{2}\right)\right] - \left(\cos^{-1}\dfrac{x}{2}\right)\dfrac{1}{2\sqrt{2x+7}}\cdot\dfrac{d}{dx}(2x+7)}{2x+7}$$

$$= \frac{\sqrt{2x+7}\,\dfrac{-1}{\sqrt{4-x^2}} - \left(\cos^{-1}\dfrac{x}{2}\right)\dfrac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)} - \frac{\cos^{-1}\dfrac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$= -\left[\frac{1}{\sqrt{4-x^2}\,\sqrt{2x+7}} + \frac{\cos^{-1}\dfrac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Question 6:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$$

Answer

Let $y = \cot^{-1}\left[\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$ ...(1)

Then, $\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}$

$= \dfrac{\left(\sqrt{1+\sin x} + \sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x} - \sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x} + \sqrt{1-\sin x}\right)}$

$= \dfrac{(1+\sin x) + (1-\sin x) + 2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x) - (1-\sin x)}$

$= \dfrac{2 + 2\sqrt{1-\sin^2 x}}{2\sin x}$

$= \dfrac{1+\cos x}{\sin x}$

$= \dfrac{2\cos^2 \dfrac{x}{2}}{2\sin \dfrac{x}{2}\cos \dfrac{x}{2}}$

$= \cot \dfrac{x}{2}$

Therefore, equation (1) becomes

$y = \cot^{-1}\left(\cot \dfrac{x}{2}\right)$

$\Rightarrow y = \dfrac{x}{2}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{d}{dx}(x)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}$

Question 7:

$(\log x)^{\log x}$, $x > 1$

Answer

Let $y = (\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x \cdot \frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Question 8:

$\cos(a\cos x + b\sin x)$, for some constant a and b.

Answer

$$\text{Let } y = \cos(a\cos x + b\sin x)$$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x)\cdot\frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x)\cdot\left[a(-\sin x) + b\cos x\right]$$

$$= (a\sin x - b\cos x)\cdot\sin(a\cos x + b\sin x)$$

Question 9:

$(\sin x - \cos x)^{(\sin x - \cos x)}, \quad \frac{\pi}{4} < x < \frac{3\pi}{4}$

Answer

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking logarithm on both the sides, we obtain

$\log y = \log \left[(\sin x - \cos x)^{(\sin x - \cos x)} \right]$

$\Rightarrow \log y = (\sin x - \cos x) \cdot \log (\sin x - \cos x)$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{y} \dfrac{dy}{dx} = \dfrac{d}{dx} \left[(\sin x - \cos x) \log (\sin x - \cos x) \right]$

$\Rightarrow \dfrac{1}{y} \dfrac{dy}{dx} = \log (\sin x - \cos x) \cdot \dfrac{d}{dx} (\sin x - \cos x) + (\sin x - \cos x) \cdot \dfrac{d}{dx} \log (\sin x - \cos x)$

$\Rightarrow \dfrac{1}{y} \dfrac{dy}{dx} = \log (\sin x - \cos x) \cdot (\cos x + \sin x) + (\sin x - \cos x) \cdot \dfrac{1}{(\sin x - \cos x)} \cdot \dfrac{d}{dx} (\sin x - \cos x)$

$\Rightarrow \dfrac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} \left[(\cos x + \sin x) \cdot \log (\sin x - \cos x) + (\cos x + \sin x) \right]$

$\therefore \dfrac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} (\cos x + \sin x) \left[1 + \log (\sin x - \cos x) \right]$

Question 10:

$x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

Answer

Let $y = x^x + x^a + a^x + a^a$

Also, let $x^x = u$, $x^a = v$, $a^x = w$, and $a^a = s$

$\therefore y = u + v + w + s$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} + \dfrac{dw}{dx} + \dfrac{ds}{dx}$ \qquad ...(1)

$u = x^x$

$\Rightarrow \log u = \log x^x$

$\Rightarrow \log u = x \log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \cdot 1 + x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x[\log x + 1] = x^x(1 + \log x) \qquad\qquad ...(2)$$

$$v = x^a$$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}(x^a)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \qquad\qquad ...(3)$$

$$w = a^x$$

$$\Rightarrow \log w = \log a^x$$

$$\Rightarrow \log w = x \log a$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w \log a$$

$$\Rightarrow \frac{dw}{dx} = a^x \log a \qquad\qquad ...(4)$$

$s = a^a$

Since a is constant, a^a is also a constant.

$$\therefore \frac{ds}{dx} = 0 \qquad\qquad ...(5)$$

From (1), (2), (3), (4), and (5), we obtain

$$\frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x \log a + 0$$

$$= x^x(1 + \log x) + ax^{a-1} + a^x \log a$$

Question 11:

$x^{x^2-3}+(x-3)^{x^2}$, for $x>3$

Answer

Let $y = x^{x^2-3}+(x-3)^{x^2}$

Also, let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

$\therefore y = u + v$

Differentiating both sides with respect to *x,* we obtain

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \qquad ...(1)$$

$u = x^{x^2-3}$

$\therefore \log u = \log\left(x^{x^2-3}\right)$

$\log u = \left(x^2 - 3\right)\log x$

Differentiating with respect to *x*, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx} = \log x \cdot \frac{d}{dx}\left(x^2-3\right) + \left(x^2-3\right)\cdot\frac{d}{dx}\left(\log x\right)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x \cdot 2x + \left(x^2-3\right)\cdot\frac{1}{x}$$

$$\Rightarrow \frac{du}{dx} = x^{x^2-3}\cdot\left[\frac{x^2-3}{x}+2x\log x\right]$$

Also,

$v = (x-3)^{x^2}$

$\therefore \log v = \log(x-3)^{x^2}$

$\Rightarrow \log v = x^2 \log(x-3)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log(x-3) \cdot \frac{d}{dx}(x^2) + x^2 \cdot \frac{d}{dx}\left[\log(x-3)\right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \log(x-3) \cdot 2x + x^2 \cdot \frac{1}{x-3} \cdot \frac{d}{dx}(x-3)$$

$$\Rightarrow \frac{dv}{dx} = v\left[2x\log(x-3) + \frac{x^2}{x-3} \cdot 1\right]$$

$$\Rightarrow \frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

Substituting the expressions of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

Question 12:

Find $\dfrac{dy}{dx}$, if $y = 12(1-\cos t), x = 10(t-\sin t), -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

Answer

It is given that, $y = 12(1-\cos t), x = 10(t-\sin t)$

$$\therefore \frac{dx}{dt} = \frac{d}{dt}\left[10(t-\sin t)\right] = 10 \cdot \frac{d}{dt}(t-\sin t) = 10(1-\cos t)$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[12(1-\cos t)\right] = 12 \cdot \frac{d}{dt}(1-\cos t) = 12 \cdot \left[0-(-\sin t)\right] = 12\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{12\sin t}{10(1-\cos t)} = \frac{12 \cdot 2\sin\dfrac{t}{2} \cdot \cos\dfrac{t}{2}}{10 \cdot 2\sin^2\dfrac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}$$

Question 13:

Find $\dfrac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}, -1 \le x \le 1$

Answer

It is given that, $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1} x + \sin^{-1} \sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1} x\right) + \frac{d}{dx}\left(\sin^{-1} \sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}}\left(-2x\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = 0$$

Question 14:

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for, $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{\left(1+x\right)^2}$$

Answer

It is given that,

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we obtain

$x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 + x^2 y = y^2 + xy^2$

$\Rightarrow x^2 - y^2 = xy^2 - x^2 y$

$\Rightarrow x^2 - y^2 = xy(y-x)$

$\Rightarrow (x+y)(x-y) = xy(y-x)$

$\therefore x+y = -xy$

$\Rightarrow (1+x)y = -x$

$\Rightarrow y = \dfrac{-x}{(1+x)}$

Differentiating both sides with respect to *x*, we obtain

$y = \dfrac{-x}{(1+x)}$

$\dfrac{dy}{dx} = -\dfrac{(1+x)\dfrac{d}{dx}(x) - x\dfrac{d}{dx}(1+x)}{(1+x)^2} = -\dfrac{(1+x)-x}{(1+x)^2} = -\dfrac{1}{(1+x)^2}$

Hence, proved.

Question 15:

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$ is a constant independent of *a* and *b*

Answer

It is given that, $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to *x*, we obtain

$$\frac{d}{dx}\Big[(x-a)^2\Big]+\frac{d}{dx}\Big[(y-b)^2\Big]=\frac{d}{dx}\big(c^2\big)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad\qquad ...(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$= -\left[\frac{(y-b)\cdot\dfrac{d}{dx}(x-a)-(x-a)\cdot\dfrac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\dfrac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\left\{\dfrac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad \left[\text{Using } (1)\right]$$

$$= -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \frac{\left[1+\dfrac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\dfrac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\dfrac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\dfrac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= \frac{\left[\dfrac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\dfrac{c^2}{(y-b)^3}} = \frac{\dfrac{c^3}{(y-b)^3}}{-\dfrac{c^2}{(y-b)^3}}$$

$= -c$, which is constant and is independent of a and b

Hence, proved.

Question 16:

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$

Answer

It is given that, $\cos y = x\cos(a+y)$

$\therefore \dfrac{d}{dx}\left[\cos y\right] = \dfrac{d}{dx}\left[x\cos(a+y)\right]$

$\Rightarrow -\sin y\dfrac{dy}{dx} = \cos(a+y)\cdot\dfrac{d}{dx}(x) + x\cdot\dfrac{d}{dx}\left[\cos(a+y)\right]$

$\Rightarrow -\sin y\dfrac{dy}{dx} = \cos(a+y) + x\cdot\left[-\sin(a+y)\right]\dfrac{dy}{dx}$

$\Rightarrow \left[x\sin(a+y) - \sin y\right]\dfrac{dy}{dx} = \cos(a+y)$ ...(1)

Since $\cos y = x\cos(a+y)$, $x = \dfrac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to

$\left[\dfrac{\cos y}{\cos(a+y)}\cdot\sin(a+y) - \sin y\right]\dfrac{dy}{dx} = \cos(a+y)$

$\Rightarrow \left[\cos y\cdot\sin(a+y) - \sin y\cdot\cos(a+y)\right]\cdot\dfrac{dy}{dx} = \cos^2(a+y)$

$\Rightarrow \sin(a+y-y)\dfrac{dy}{dx} = \cos^2(a+b)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\cos^2(a+b)}{\sin a}$

Hence, proved.

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

Answer

It is given that, $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$

$\therefore \dfrac{dx}{dt} = a \cdot \dfrac{d}{dt}(\cos t + t \sin t)$

$\qquad = a\left[-\sin t + \sin t \cdot \dfrac{d}{dx}(t) + t \cdot \dfrac{d}{dt}(\sin t)\right]$

$\qquad = a[-\sin t + \sin t + t \cos t] = at \cos t$

$\dfrac{dy}{dt} = a \cdot \dfrac{d}{dt}(\sin t - t \cos t)$

$\qquad = a\left[\cos t - \left\{\cos t \cdot \dfrac{d}{dt}(t) + t \cdot \dfrac{d}{dt}(\cos t)\right\}\right]$

$\qquad = a\left[\cos t - \{\cos t - t \sin t\}\right] = at \sin t$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \dfrac{at \sin t}{at \cos t} = \tan t$

Then, $\dfrac{d^2 y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(\tan t) = \sec^2 t \cdot \dfrac{dt}{dx}$

$\qquad = \sec^2 t \cdot \dfrac{1}{at \cos t} \qquad \left[\dfrac{dx}{dt} = at \cos t \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{at \cos t}\right]$

$\qquad = \dfrac{\sec^3 t}{at}, \; 0 < t < \dfrac{\pi}{2}$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x, and find it.

Answer

It is known that, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when $x \geq 0$, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When $x < 0$, $f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

Question 19:

Using mathematical induction prove that $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

Answer

To prove: $P(n): \dfrac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n

For $n = 1$,

$$P(1): \dfrac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

∴P(n) is true for n = 1

Let P(k) is true for some positive integer k.

That is, $P(k): \dfrac{d}{dx}(x^k) = kx^{k-1}$

It has to be proved that P(k + 1) is also true.

Consider $\dfrac{d}{dx}(x^{k+1}) = \dfrac{d}{dx}(x \cdot x^k)$

$\qquad\qquad = x^k \cdot \dfrac{d}{dx}(x) + x \cdot \dfrac{d}{dx}(x^k)$ \qquad [By applying product rule]

$\qquad\qquad = x^k \cdot 1 + x \cdot k \cdot x^{k-1}$

$\qquad\qquad = x^k + kx^k$

$\qquad\qquad = (k+1) \cdot x^k$

$\qquad\qquad = (k+1) \cdot x^{(k+1)-1}$

Thus, P(k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for

every positive integer n.

Hence, proved.

Question 20:

Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain

the sum formula for cosines.

Answer

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right] = \frac{d}{dx}(\sin A \cos B) + \frac{d}{dx}(\cos A \sin B)$$

$$\Rightarrow \cos(A+B) \cdot \frac{d}{dx}(A+B) = \cos B \cdot \frac{d}{dx}(\sin A) + \sin A \cdot \frac{d}{dx}(\cos B)$$

$$+ \sin B \cdot \frac{d}{dx}(\cos A) + \cos A \cdot \frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B) \cdot \frac{d}{dx}(A+B) = \cos B \cdot \cos A \frac{dA}{dx} + \sin A(-\sin B)\frac{dB}{dx}$$

$$+ \sin B(-\sin A) \cdot \frac{dA}{dx} + \cos A \cos B \frac{dB}{dx}$$

$$\Rightarrow \cos(A+B) \cdot \left[\frac{dA}{dx} + \frac{dB}{dx}\right] = (\cos A \cos B - \sin A \sin B) \cdot \left[\frac{dA}{dx} + \frac{dB}{dx}\right]$$

$$\therefore \cos(A+B) = \cos A \cos B - \sin A \sin B$$

Question 22:

If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y = (mc - nb) f(x) - (lc - na) g(x) + (lb - ma) h(x)$$

Then, $\dfrac{dy}{dx} = \dfrac{d}{dx}\big[(mc - nb) f(x)\big] - \dfrac{d}{dx}\big[(lc - na) g(x)\big] + \dfrac{d}{dx}\big[(lb - ma) h(x)\big]$

$$= (mc - nb) f'(x) - (lc - na) g'(x) + (lb - ma) h'(x)$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus, $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Question 23:

If $y = e^{a\cos^{-1} x}, -1 \le x \le 1$, show that $\left(1 - x^2\right)\dfrac{d^2 y}{dx^2} - x\dfrac{dy}{dx} - a^2 y = 0$

Answer

It is given that, $y = e^{a\cos^{-1} x}$

Taking logarithm on both the sides, we obtain

$\log y = a \cos^{-1} x \log e$

$\log y = a \cos^{-1} x$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{y}\dfrac{dy}{dx} = a \times \dfrac{-1}{\sqrt{1-x^2}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-ay}{\sqrt{1-x^2}}$

By squaring both the sides, we obtain

$\left(\dfrac{dy}{dx}\right)^2 = \dfrac{a^2 y^2}{1-x^2}$

$\Rightarrow \left(1-x^2\right)\left(\dfrac{dy}{dx}\right)^2 = a^2 y^2$

$\left(1-x^2\right)\left(\dfrac{dy}{dx}\right)^2 = a^2 y^2$

Again differentiating both sides with respect to x, we obtain

$\left(\dfrac{dy}{dx}\right)^2 \dfrac{d}{dx}\left(1-x^2\right) + \left(1-x^2\right) \times \dfrac{d}{dx}\left[\left(\dfrac{dy}{dx}\right)^2\right] = a^2 \dfrac{d}{dx}\left(y^2\right)$

$\Rightarrow \left(\dfrac{dy}{dx}\right)^2 (-2x) + \left(1-x^2\right) \times 2 \dfrac{dy}{dx}.\dfrac{d^2 y}{dx^2} = a^2 .2y.\dfrac{dy}{dx}$

$\Rightarrow \left(\dfrac{dy}{dx}\right)^2 (-2x) + \left(1-x^2\right) \times 2 \dfrac{dy}{dx}.\dfrac{d^2 y}{dx^2} = a^2 .2y.\dfrac{dy}{dx}$

$\Rightarrow -x\dfrac{dy}{dx} + \left(1-x^2\right)\dfrac{d^2 y}{dx^2} = a^2 .y$ $\left[\dfrac{dy}{dx} \neq 0\right]$

$\Rightarrow \left(1-x^2\right)\dfrac{d^2 y}{dx^2} - x\dfrac{dy}{dx} - a^2 y = 0$

Hence, proved.